

SELF-SIMILAR GROUPS ACTING ESSENTIALLY FREELY ON THE BOUNDARY OF THE BINARY ROOTED TREE

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ABSTRACT. We completely classify groups generated by 3-state automata over 2-letter alphabet that act essentially freely on the boundary of the binary rooted tree.

1. INTRODUCTION

Groups generated by Mealy type automata represent an important and interesting class of groups with connections to different branches of mathematics, such as dynamical systems (including symbolic dynamics and holomorphic dynamics), computer science, topology and probability. For more details about this class of groups we refer the reader to survey papers [GNS00, BS10].

In the whole class of groups generated by automata, there is an important subclass of self-similar groups. These are the groups generated by initial Mealy type automata that are determined by all states of a non initial automaton. The natural characteristic of such groups, which we will call *complexity*, is the pair (m, n) of two integers, $m \geq 2, n \geq 2$, where m is a number of states and n is a cardinality of the alphabet. There are 6 groups of complexity $(2, 2)$ and the “largest” (most complicated) of them is the lamplighter group $\mathcal{L} = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. It is shown in [BGK⁺08] and [Mun09] that there is not more than 115 different (up to isomorphism) groups of complexity $(3, 2)$, although the number of corresponding automata up to certain natural symmetry is 194. Even though the complete characterization of $(3, 2)$ -groups is not achieved yet, a lot of information about these groups has been obtained. The motivation for this paper is twofold: partially it comes from the necessity to understand this class of groups better, and additionally, it represents the venture in the search of new interesting examples of self-similar groups that might potentially serve as answers to questions posed at the end of the paper.

The problem of characterization of groups of complexity (m, n) becomes extremely difficult when either of parameters m or n go to infinity as the number of corresponding automata grows very rapidly and the involved groups become more and more complicated. Even in the case of complexity $(3, 2)$ there are groups with quite unusual properties like being branch (the notion introduced in [Gri00a], see also [Gri00b]) or being amenable but not subexponentially amenable [GŻ02, BV05]. There are groups of complexity $(5, 2)$ and $(4, 2)$ of intermediate (between exponential and polynomial) growth [Gri84, BP06].

Groups generated by finite automata defined over the m -letter alphabet, in particular self-similar groups, naturally act on the m -regular rooted tree $T = T_m$ (m a cardinality of alphabet) and on its boundary, which topologically is homeomorphic to the Cantor set. This action preserves the uniform Bernoulli measure μ on the

boundary. Therefore one can study a topological dynamical system $(G, \partial T)$ or metric dynamical system $(G, \partial T, \mu)$. Ergodicity of the latter is equivalent to the level transitivity of the action of G on T .

The important class of actions are topologically free actions and essentially free actions. For the first case the assumption is that for each nonidentity element $g \in G$ the set of fixed points $Fix(g)$ is meager (i.e. can be represented as a countable union of nowhere dense sets). For the second case the assumption is that for any nonidentity element g of a group the measure of a fixed point set of g is zero. These types of actions play especially important role in various studies in dynamical systems, operator algebras, and modern directions of group theory like theory of cost or rank gradient [Gab10, AN12]. Self-similar groups acting essentially freely on ∂T can potentially be used to construct new examples of scale-invariant groups [NP11], and have connection to the class of hereditary just-infinite groups [Gri00b].

The opposite to the notion of a free action are totally nonfree actions considered recently in [Ver11, Gri11]. These are the actions, for which stabilizers of different points of the set of full measure are different. Surprisingly many groups generated by finite automata, in particular those of them that are branch or weakly branch, act totally nonfree. Totally non free actions are also important for the theory of operator algebras and for rapidly developing now theory of invariant random subgroups [Ver11, AGV12, Bow12, BGK12].

The goal of this paper is to describe all $(3, 2)$ -groups acting freely on the boundary. Although in general for group actions on topological spaces with invariant measure there is no connection between topological freeness and essential freeness for groups generated by finite automata acting on a boundary of a tree (in a way prescribed by determining automaton) these two notions are equivalent as observed by Kambites, Silva and Steinberg in [KSS06].

To each $(3, 2)$ -automaton one assigns a unique number from 1 to 5832 according to certain natural lexicographic order on the set of all these automata (see Section 2 and [BGK⁺08]). Obviously, two automata whose minimizations can be obtained from each other by permuting the states, letters, or passing to the inverse automaton, generated isomorphic groups whose actions on ∂T_2 are conjugate. This defines an equivalence relation on the set of all automata that we call *minimal symmetry*. By construction of this relation, for each equivalence class it is enough to study only one representative. In the main theorem below we list all groups generated by $(3, 2)$ -automata acting essentially freely on ∂T_2 and for each group we give in brackets the numbers of representatives of equivalence classes of automata that generate this group.

Our main result is:

Theorem 1.1. *Among all groups generated by 3-state automata over 2-letter alphabet the only groups that act essentially freely on the boundary of the tree T_2 are the following:*

- Trivial group [1];
- Group $\mathbb{Z}/2\mathbb{Z}$ of order 2 [1090, 1094];
- Klein group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ [730, 734, 766, 770, 774, 2232, 2264, 2844, 2880];
- $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ [802, 806, 810, 2196, 2260];
- Infinite cyclic group \mathbb{Z} [730, 734, 766, 770, 774, 2232, 2264, 2844, 2880];

- \mathbb{Z}^2 [771,803,807];
- Infinite dihedral group D_∞ [820,824,865,919,928,932,936,2226,2358,2394,2422,2874];
- Baumslag-Solitar group $BS(1, 3)$ [870,924]
- Baumslag-Solitar group $BS(1, -3)$ [2294,2320]
- Extension $((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ of the lamplighter group by $\mathbb{Z}/2\mathbb{Z}$ [891];
- Free group F_3 of rank 3 generated by the Aleshin automaton [2240]
- Free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ of three groups of order 2 generated by Bellaterra automaton [846]
- Lamplighter group $\mathcal{L} \cong (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ [821,839,930,2374,2388];
- Extension $\mathbb{Z}^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ of the \mathbb{Z}^2 group by $\mathbb{Z}/2\mathbb{Z}$ [2277,2313,2426];
- Metabelian group $((\frac{1}{2}\mathbb{Z} \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}) \rtimes (\mathbb{Z}/2\mathbb{Z})) \rtimes \mathbb{Z}$ [2372];
- Extension $((\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ of a rank 2 lamplighter group $\mathcal{L}_{2,2} \cong (\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$ [2193],

where the numbers in brackets indicate corresponding numbers of (3,2)-automata defined in Section 2.

Note that the notation $\mathcal{L}_{2,2}$ used in the above theorem is borrowed from [GK12], where $\mathcal{L}_{p,n}$ denotes the group $(\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z}$ called the rank n lamplighter group. We also denote throughout the paper by \mathcal{L} the “standard” lamplighter group $\mathcal{L}_{2,1}$.

The paper is organized as follows. In Section 2 we recall main definitions from a theory of groups generated by automata, and introduce necessary notation related to the class of 3-state automata over 2-letter alphabet. Section 3 discusses various types of free actions and lists relevant results in this area. The main Theorem 1.1 is proved in Section 4. Finally, we conclude the paper with open questions in Section 5.

2. GROUPS GENERATED BY AUTOMATA AND CLASSIFICATION NOTATIONS

In this section we remind the main notions related to automaton groups and to the problem of classification of (3,2)-groups.

Let X be a finite set of cardinality d and let X^* denote the free monoid generated by X , which consists of finite words over X . This monoid has a natural structure of a rooted tree $T(X)$ (or T_d as $|X| = d$) in which v is adjacent to vx for any $v \in X^*$ and $x \in X$. The empty word corresponds to the root of the tree and X^n corresponds to the n -th level of the tree. We will be interested in the groups of automorphisms and semigroups of endomorphisms of X^* . Any such homomorphism can be defined via the notion of initial automaton.

Definition 1. A Mealy automaton (or simply automaton) is a tuple (Q, X, π, λ) , where Q is a set of states, X is a finite alphabet, $\pi: Q \times X \rightarrow Q$ is a transition function and $\lambda: Q \times X \rightarrow X$ is an output function. If the set of states Q is finite the automaton is called finite. If for every state $q \in Q$ the output function $\lambda(q, x)$ induces a permutation of X , the automaton \mathcal{A} is called invertible. Selecting a state $q \in Q$ produces an initial automaton \mathcal{A}_q .

Automata are often represented by the *Moore diagrams*. The Moore diagram of an automaton $\mathcal{A} = (Q, X, \pi, \lambda)$ is a directed graph in which the vertices are the elements of Q and the edges have form $q \xrightarrow{x|\lambda(q,x)} \pi(q, x)$ for $q \in Q$ and $x \in X$. If the automaton is invertible, then it is convenient to label vertices of the Moore diagram

by the permutation $\lambda(q, \cdot)$ and leave just first components from the labels of the edges. An example of Moore diagram is shown in Figure 2.

Any initial automaton induces a homomorphism of X^* . Given a word $v = x_1x_2x_3 \dots x_n \in X^*$ it scans its first letter x_1 and outputs $\lambda(x_1)$. The rest of the word is handled in a similar fashion by the initial automaton $\mathcal{A}_{\pi(x_1)}$. Formally speaking, the functions π and λ can be extended to $\pi: Q \times X^* \rightarrow Q$ and $\lambda: Q \times X^* \rightarrow X^*$ via

$$\begin{aligned}\pi(q, x_1x_2 \dots x_n) &= \pi(\pi(q, x_1), x_2x_3 \dots x_n), \\ \lambda(q, x_1x_2 \dots x_n) &= \lambda(q, x_1)\lambda(\pi(q, x_1), x_2x_3 \dots x_n).\end{aligned}$$

By construction, any initial automaton acts on X^* (viewed as a tree) as an endomorphism. In the case of invertible automaton it acts as an automorphism.

Definition 2. *The semigroup (group) generated by all states of automaton \mathcal{A} is called the automaton semigroup (automaton group) and denoted by $S(\mathcal{A})$ (respectively $G(\mathcal{A})$).*

Note, that the composition and the inverse of transformations defined by (finite) automata are again defined by (finite) automata. For example, the *inverse automaton* \mathcal{A}_q^{-1} to automaton \mathcal{A}_q defining the inverse of the transformation of X^* defined by \mathcal{A}_q is obtained from \mathcal{A}_q simply by flipping the components of the labels of all edges in its Moore diagram.

Another popular name for automaton groups and semigroups is self-similar groups and semigroups (see [Nek05]). Among general properties of these groups we will use that all of them are residually finite, and thus Hopfian by Malcev's theorem [Mal40], i.e. each surjective endomorphism of an automaton group on itself is an isomorphism.

We will need a notion of a *section* of a homomorphism at a vertex of the tree. Let g be a homomorphism of the tree X^* and $x \in X$. Then for any $v \in X^*$ we have

$$g(xv) = g(x)v'$$

for some $v' \in X^*$. The map $g|_x: X^* \rightarrow X^*$ given by

$$g|_x(v) = v'$$

defines an endomorphism of X^* and is called the *section* of g at vertex x . Furthermore, for any $x_1x_2 \dots x_n \in X^*$ we define

$$g|_{x_1x_2 \dots x_n} = g|_{x_1}|_{x_2} \dots |_{x_n}.$$

Given an endomorphism g of X^* one can construct an initial automaton $\mathcal{A}(g)$ whose action on X^* coincides with that of g as follows. The set of states of $\mathcal{A}(g)$ is the set $\{g|_v: v \in X^*\}$ of different sections of g at the vertices of the tree. The transition and output functions are defined by

$$\begin{aligned}\pi(g|_v, x) &= g|_{vx}, \\ \lambda(g|_v, x) &= g|_v(x).\end{aligned}$$

Throughout the paper we will use the following convention. If g and h are the elements of some (semi)group acting on set A and $a \in A$, then

$$(1) \quad gh(a) = h(g(a)).$$

In particular, this means that we consider right action of $\text{Sym}(X)$ on X .

Taking into account convention (1) one can compute sections of any element of an automaton semigroup as follows. If $g = g_1 g_2 \cdots g_n$ and $v \in X^*$, then

$$(2) \quad g|_v = g_1|_v \cdot g_2|_{g_1(v)} \cdots g_n|_{g_1 g_2 \cdots g_{n-1}(v)}.$$

For any automaton group G there is a natural embedding

$$G \hookrightarrow G \wr \text{Sym}(X)$$

defined by

$$(3) \quad G \ni g \mapsto (g|_0, g|_1, \dots, g|_{d-1})\lambda(g) \in G \wr \text{Sym}(X),$$

where $g|_0, g|_1, \dots, g|_{d-1}$ are the sections of g at the vertices of the first level, and $\lambda(g)$ is a permutation of X induced by the action of g on the first level of the tree.

The above embedding is convenient in computations involving the sections of automorphisms, as well as for defining automaton groups. We will call it the *wreath recursion* defining the group.

The following important notions related to groups generated by automata will be used throughout the text.

Definition 3. A self-similar group G is self-replicating if, for every vertex $u \in X^*$, the homomorphism $\phi_u: \text{Stab}_G(u) \rightarrow G$ from the stabilizer of the vertex u in G to G , given by $\phi_u(g) = g|_u$, is surjective.

Definition 4. We say that an element g of a self-similar group (resp., a self-similar group G) acts spherically transitively, if g (resp., G) acts transitively on each level X^n of the tree X^* .

Note, that a self-similar groups acting on binary tree is infinite if and only if it acts spherically transitively (see Lemma 3 in [BGK⁺08]).

An important class of groups acting on trees is the class of branch groups [Gri00b, BGŠ03].

Definition 5. Let G be a group acting on the rooted tree X^* .

- The rigid stabilizer of a vertex $v \in X^*$ in G is a subgroup $\text{Rist}_G(v)$ of G that consists of elements that act nontrivially only on the vertices that have v as a prefix.
- The rigid stabilizer of a level n of X^* in G is a subgroup $\text{Rist}_G(n)$ of G that is generated by rigid stabilizers of all the vertices of this level.

Definition 6. A group G acting on the rooted tree X^* is called

- weakly branch, if for each $n \geq 1$ the rigid stabilizer $\text{Rist}_n(G)$ of the n -th level of X^* is nontrivial;
- branch, if for each $n \geq 1$ the rigid stabilizer $\text{Rist}_n(G)$ of the n -th level of X^* has finite index in G .

Further, we will need a notion of a dual automaton $\hat{\mathcal{A}}$ to automaton \mathcal{A} , which is obtained from \mathcal{A} by “switching the roles” of states and letters of the alphabet. The formal definition is given below.

Definition 7. Given a finite automaton $\mathcal{A} = (Q, X, \pi, \lambda)$ its dual automaton $\hat{\mathcal{A}}$ is a finite automaton $(X, Q, \hat{\lambda}, \hat{\pi})$, where

$$\begin{aligned}\hat{\lambda}(x, q) &= \lambda(q, x), \\ \hat{\pi}(x, q) &= \pi(q, x)\end{aligned}$$

for any $x \in X$ and $q \in Q$.

Note that the dual of the dual of an automaton \mathcal{A} coincides with \mathcal{A} . The semigroup $\mathbb{S}(\hat{\mathcal{A}})$ generated by dual automaton $\hat{\mathcal{A}}$ of automaton \mathcal{A} acts on the free monoid Q^* . This action induces the action on $\mathbb{S}(\mathcal{A})$. Similarly, $\mathbb{S}(\mathcal{A})$ acts on $\mathbb{S}(\hat{\mathcal{A}})$.

Definition 8. For an automaton semigroup G generated by automaton \mathcal{A} the dual semigroup \hat{G} to G is a semigroup generated by a dual automaton $\hat{\mathcal{A}}$.

A particularly important class of automata is the class of bireversible automata as they give rise to interesting examples of groups, provide an approach to prove freeness properties, and admit solutions to certain algorithmical problems [GM05, SV11, AKL⁺11, Kli12].

Definition 9. An automaton \mathcal{A} is called bireversible if it is invertible, its dual is invertible, and the dual to \mathcal{A}^{-1} are invertible.

Now we describe shortly the notation and some basic facts used in the classification of $(3, 2)$ -groups [BGK⁺08].

Every 3-state automaton \mathcal{A} with set of states $S = \{0, 1, 2\}$ acting on the 2-letter alphabet $X = \{0, 1\}$ is assigned a unique number as follows. Given the wreath recursion

$$\begin{cases} \mathbf{0} = (a_{11}, a_{12})\sigma^{a_{13}}, \\ \mathbf{1} = (a_{21}, a_{22})\sigma^{a_{23}}, \\ \mathbf{2} = (a_{31}, a_{32})\sigma^{a_{33}}, \end{cases}$$

defining the automaton \mathcal{A} , where $a_{ij} \in \{0, 1, 2\}$ for $j \neq 1$ and $a_{i3} \in \{0, 1\}$, $i = 1, 2, 3$, assign the number

$$\begin{aligned}\text{Number}(\mathcal{A}) &= \\ & a_{11} + 3a_{12} + 9a_{21} + 27a_{22} + 81a_{31} + \\ & 243a_{32} + 729(a_{13} + 2a_{23} + 4a_{33}) + 1\end{aligned}$$

to \mathcal{A} . With this agreement every $(3, 2)$ -automaton a unique number in the range from 1 to 5832 is assigned. The numbering of the automata is induced by the lexicographic ordering of tuples $(a_{11}, a_{12}, \dots, a_{33})$ that define all automata in the class. Each of the automata numbered 1 through 729 generates the trivial group, since all vertex permutations are trivial in this case. Each of the automata numbered 5104 through 5832 generates the cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order 2, since both states represent the automorphism that acts by changing all letters in every word over X . Therefore the nontrivial part of the classification is concerned with the automata numbered by 730 through 5103.

Denote by \mathcal{A}_n the automaton numbered by n and by G_n the corresponding group $\mathbb{G}(\mathcal{A}_n)$ of tree automorphisms. Sometimes we will use just the number to refer to the corresponding automaton or group.

The following three operations on automata do not change the isomorphism class of the group generated by the corresponding automaton (and do not change the action on the tree up to conjugation):

- (i) passing to inverses of all generators (equivalently, passing to the inverse automaton),
- (ii) permuting the states of the automaton,
- (iii) permuting the alphabet letters.

Definition 10. *Two automata \mathcal{A} and \mathcal{B} that can be obtained from one another by using a composition of the operations (i)–(iii), are called symmetric.*

Additional identifications can be made after automata minimization is applied. Recall, that the minimization of an automaton is a standard procedure (see, for example, [Eil76]) that identifies the states that induce identical transformations of X^* .

Definition 11. *If the minimization of an automaton \mathcal{A} is symmetric to the minimization of an automaton \mathcal{B} , we say that the automata \mathcal{A} and \mathcal{B} are minimally symmetric and write $\mathcal{A} \sim \mathcal{B}$.*

There are 194 classes of $(3, 2)$ -automata that are pairwise not minimally symmetric. At present, it is known that there are no more than 115 non-isomorphic $(3, 2)$ -automaton groups and all these groups are listed in [BGK⁺08, Mun09].

In this paper, since we are looking for essentially free actions of groups, we will actually distinguish non minimally symmetric automata generating isomorphic groups, as the same group may have different actions on ∂T_2 . So we will work with all 194 classes of not minimally symmetric automata.

3. TYPES OF ACTIONS AND MAIN TOOLS

There are different ways to define the freeness of a group actions. The definition below works in the general context of arbitrary topological (or, respectively, measure) space, but we will work only in the context of the actions of self-similar groups on the boundary ∂T of the rooted tree T . Recall, that ∂T consists of all infinite paths without backtracking initiating from the root (equivalently, ∂X^* can be thought of as the set of all infinite words over X). The set ∂T is endowed with a topology (in, fact, it is an ultrametric space) in which two paths are declared to be close if they have long common beginning. With this topology it is homeomorphic to the Cantor set. Further, one can define a uniform Bernoulli measure on ∂T making this space a measure space.

Now we remind the general definition and set up some notation. Let G be a countable group acting on a complete metric space Y . Denote by Y_- the set of points with nontrivial stabilizer and by Y_+ the set of points with trivial stabilizer.

Definition 12.

- (1) *The action (G, Y) is said to be absolutely free if all points have trivial stabilizers.*
- (2) *The action (G, Y) is topologically free if Y_- is a meager set (i.e., it can be represented as a countable union of nowhere dense sets).*

- (3) Suppose that the action (G, Y) has a G -invariant (not necessarily finite) Borel measure μ . The action on the measure space (G, Y, μ) is said to be essentially free if $\mu(Y_-) = 0$.

We consider this definition in the context of self-similar groups acting on the boundary $\partial T(X)$ of corresponding tree. This gives immediately topological dynamical system $(G, \partial T(X))$. As mentioned above, $\partial T(X)$ can be considered as a measure space with a uniform Bernoulli measure, which enables us to talk about the essential freeness of the action of G on $\partial T(X)$. An important result here is that in the case of groups generated by finite state automata the notions of topological freeness and essential freeness are identical according to the following two propositions.

Proposition 3.1 ([Gri11], Corollary 4.3). *A spherically transitive essentially free action on the boundary of a tree is topologically free.*

Proposition 3.2 ([KSS06], Theorem 4.2.). *For groups generated by finite automata, any topologically free action is essentially free.*

We note, that the terminology used in [KSS06, SVV11] is somewhat different from the one used here. For example, the topological freeness bears the name of freeness in the sense of Baire category, and the essential freeness is referred to as freeness in the sense of ergodic theory. Further, the definitions used for these types of freeness are different, but equivalent in the case of countable groups (in which we are interested anyway). Namely, if for $g \in G$ one denotes by $\text{Fix}(g)$ the subset of X fixed by g , then we have

$$X_- = \bigcup_{g \in G} \text{Fix}(g).$$

Therefore, if G is countable, one can replace the condition that X_- has measure zero (resp., X_- is meager) by the equivalent condition that $\text{Fix}(g)$ has measure zero (resp., meager) for each nonidentity $g \in G$.

In order to establish that a group does not act topologically (and essentially) freely, one can just find an element $g \in G$ and a vertex $v \in X^*$ fixed by g such that $g|_v$ is trivial (because in this case all points in the cylindrical set c_v , which is open (and has positive measure) will have g in their stabilizers.

Definition 13. *For a vertex $v \in X^*$ the set of all $g \in G$ that fix v and such that $g|_v$ is trivial forms a subgroup $\text{triv}_G(v)$ of G called the trivializer of v .*

Definition 14. *The action of a group G on a rooted tree is called locally nontrivial if trivializers of all vertices of the tree are trivial.*

As observed above, if the action is not locally trivial, it cannot be topologically or essentially free. It is not hard to prove the converse in the case of countable group and topological freeness.

Proposition 3.3 ([Gri11], Proposition 4.2.). *The action of a countable group on the boundary of a tree is topologically free if and only if it is locally nontrivial.*

This observation constitutes one of the main tools to determine that a self-similar group *does not* act essentially freely on the boundary of a tree. Of course, one can simply apply a brute force to find such an element, but in case of self-replicating

groups it can be made almost automatic in many cases by using the the following procedure.

Suppose $G = \mathbb{G}(\mathcal{A})$ is a group generated by automaton \mathcal{A} with states a_1, a_2, \dots, a_n . With a slight abuse of notation we will treat a_i 's as generators of G and write $G = \langle a_1, a_2, \dots, a_n \rangle$. First, we calculate the finite generating set $\{s_j, j \in J\}$ of the stabilizer of the first level of the tree $\text{Stab}_G(1)$ in G . This is a subgroup of finite index and a Reidemeister-Schreier procedure can be used for that [MKS04].

Let F_A denote the free group generated by elements a_1, a_2, \dots, a_n . The wreath recursion that defines an automaton induces an embedding

$$F_A \hookrightarrow F_A \wr \text{Sym}(X)$$

defined by

$$(4) \quad F_A \ni g \mapsto (g|_0, g|_1, \dots, g|_{d-1})\lambda(g) \in F_A \wr \text{Sym}(X).$$

With a slight abuse of notation, we will denote by s_j also a word over $A \cup A^{-1}$ in F_A that is mapped to $s_j \in G$ under the canonical epimorphism $F_A \rightarrow G$. Then we decompose each $s_j \in F_A$ as a pair $(s_j|_0, s_j|_1) \in F_A \times F_A$ using the wreath recursion embedding (4). The first components $s_j|_0$ of above pairs generate a subgroup H of F_A . After applying the Nielsen reduction to the generators of this subgroup, keeping track of second coordinates, we obtain the generating set of $\langle (s_j|_0, s_j|_1), j \in J \rangle < F_A \times F_A$ whose projection onto the first coordinate is Nielsen reduced [LS01]:

$$(5) \quad t_1 = (b_1, w_1), \dots, t_l = (b_m, w_m), \quad t_{m+1} = (1, r_1), \dots, t_{m+l} = (1, r_l),$$

where $\{b_1, \dots, b_m\}$ is a Nielsen reduced generating set for H , $w_i \in F_A$ and $m+l = |J|$. We will call such a representation for $\text{Stab}_G(1)$ the *Mikhailova system* for G . The reason for such name is explained below.

If any of r_i , $i = 1, \dots, l$ represents a nonidentity element of G , then the corresponding pair $(1, r_i)$ will represent a nonidentity element of G that belongs to the trivializer of vertex 1. Thus, the action of G on ∂T_2 would not be essentially free.

Showing that the group actually does act essentially freely is usually much harder, as witnessed by the last two sections. The main tool here is the Proposition 3.4 below. This proposition is similar to Proposition 3.3, but it additionally uses self-similarity of a group. Recall that the notion of a rigid stabilizer was introduced in Definition 5.

Proposition 3.4 ([Gri11], Proposition 4.5.). *For a group G generated by finite automaton, acting on a binary tree T_2 , the action on ∂T_2 is essentially free if and only if the rigid stabilizer of the first level $\text{Rist}_G(1)$ is trivial.*

The problem is that it is harder to show that the rigid stabilizer is trivial, than to find an element witnessing its nontriviality. The main method here is based on finding the presentation of a group by generators and relators. Note, that for a non-binary tree the condition of local nontriviality cannot be formulated in terms of rigid stabilizers.

We now go back to Equations (5). In the case when H coincides with F_A , which is the case when G is self-replicating, we get $m = n$ and this equation is transformed to (after reordering the generators, if necessary):

$$t_1 = (a_1, w_1), \dots, t_l = (a_n, w_n), \quad t_{n+1} = (1, r_1), \dots, t_{n+l} = (1, r_l).$$

We can further assume that all r_i 's represent the identity element in G (otherwise, as stated above, the action of G is not essentially free). Suppose additionally that

$$\langle w_1, w_2, \dots, w_n \rangle = F_A.$$

Then the map $\phi: a_i \rightarrow w_i$ extends to an automorphism of F_A . In this case we say that the presentation of the group G by a finite automaton belongs to the *diagonal type*. This condition does not depend on how the pairs of elements are reduced by the Nielsen transformations. Note, that the case when ϕ is the identity automorphism, one obtains a subgroup of $F_A \times F_A$ that was used by Mikhailova in [Mih58] to prove that the membership problem for direct products of free groups is algorithmically unsolvable. This is why we attribute this notion to Mikhailova.

The following proposition follows immediately from Proposition 3.4.

Proposition 3.5 ([Gri11], Proposition 5.1). *Suppose that G is a group generated by finite automaton acting on a binary tree and having the first-level stabilizer that can be reduced by the Nielsen transformations to the diagonal type. Let ϕ be the above-constructed automorphism of the free group F_A . Then the action is essentially free if and only if ϕ induces an automorphism of the group G .*

For some groups we use the following useful proposition that allows us to establish essential freeness of the action in the case of groups generated by finite bireversible automata, i.e. invertible automata, whose dual, and dual to the inverse are invertible as well.

Proposition 3.6 ([SVV11], Corollary 2.10). *A group generated by a bireversible automaton acts topologically and essentially freely on the boundary of the tree.*

In the end of this section we would like to bring the attention to the connection between groups acting essentially freely on ∂T_2 and other classes of groups. Namely, we prove that each hereditary just infinite self-similar group acts essentially freely on ∂T_d , and that the essentially free groups could be used to create new examples of scale-invariant groups. We start from definitions.

Definition 15. *The group G is called just infinite if it is infinite, but each proper quotient of G is finite.*

Definition 16 ([Gri00b]). *The group G is called hereditary just infinite if each finite index subgroup of G is just-infinite.*

Note that both hereditary just infinite groups and branch groups play a crucial role in the trichotomy classifying finitely generated just-infinite groups [Gri00b]. According to this trichotomy any finitely generated just-infinite group is either a branch group or can easily be constructed from a simple group or from a hereditarily just-infinite group.

Proposition 3.7. *Each hereditary just infinite self-similar group G generated by (possibly infinite) automaton over alphabet X that acts transitively on the first level of X^* , acts essentially freely on $\partial T_{|X|}$.*

Note that in the case of binary tree ($|X| = 2$) the condition of transitivity of G on the first level of X^* is satisfied automatically.

Proof. Suppose the action on $\partial T_{|X|}$ is not essentially free. Then by Proposition 3.3 there is a nonidentity element g and a vertex $v \in X^*$ fixed by g with $g|_v = 1$. By self-similarity, we may assume that v is a vertex of the first level.

For each $w \in X^*$ let T_w denote the tree hanging down from the vertex w and let $M_w = \text{Stab}_G(w)|_{T_w}$ be the group consisting of all sections of elements of $\text{Stab}_G(w)$ at vertex w . Since G acts transitively on the first level of X^* , all groups M_w for $w \in X^1$ are conjugate. In particular, they are either all finite or all infinite. On the other hand, if all of M_w , $w \in X^1$ are finite, then $\text{Stab}_G(1)$ must be finite as it embeds into $\prod_{w \in X^1} M_w$ via

$$\text{Stab}_G(1) \ni g \mapsto (g|_1, g|_2, \dots, g|_{|X|}) \in \prod_{w \in X^1} M_w.$$

Since G is infinite and $\text{Stab}_G(1)$ has finite index in G , we conclude that M_v is infinite. Now consider an epimorphism

$$\psi: \text{Stab}_G(1) \rightarrow M_v$$

defined by $\psi(g) = g|_v$. Since M_v is infinite and ψ is onto, the kernel of ψ has an infinite index in $\text{Stab}_G(1)$, contradicting to the fact that $\text{Stab}_G(1)$ is just infinite, which must be the case as G is hereditary just infinite and $\text{Stab}_G(1)$ has a finite index in G . \square

We note, however, that there is currently no known examples of hereditary just-infinite self-similar groups. In view of this the above proposition tells that we have to look for such examples in the class of groups that act essentially freely on the boundary of the tree (see Question 5 in Section 5).

Recall, that a group G is called *B-scale-invariant* if there is a sequence of finite index subgroups of G that are all isomorphic to G and whose intersection is a finite group. This class was introduced by Benjamini (this is why we add “B” in front of “scale-invariant”) and we refer the reader to the most recent paper discussing this class is [NP11] for details. We call a group *scale-invariant* if there is a sequence of sequence of finite index subgroups of G that are all isomorphic to G and whose intersection is trivial.

Proposition 3.8. *A self-similar self-replicating group acting essentially freely on $\partial T(X)$ is scale invariant.*

Proof. Let G be as described in the statement. Then for each vertex $u \in X^*$ consider the stabilizer $\text{Stab}_G(u)$ of u in G . First of all, the index of $\text{Stab}_G(u)$ cannot exceed $|X|^{|u|}$ as vertex u cannot be moved by G outside its level, which has $|X|^{|u|}$ vertices. Since G is self-replicating, the canonical homomorphism $\phi_u: \text{Stab}_G(u) \rightarrow G$ defined by $\phi_u(g) = g|_u$ is surjective. On the other hand, the kernel of this homomorphism is trivial since otherwise we would obtain a nonidentity element in the trivializer of u in G contradicting to the essential freeness of the action of G on $\partial T(X)$ by Proposition 3.3. Therefore, $\text{Stab}_G(u)$ is isomorphic to G .

Since the action of G on $\partial T(X)$ is essentially free, the set of points in ∂T_2 that have trivial stabilizers in G has full measure. Let $\omega \in \partial T(X)$ be a point in this

set, so that $\text{Stab}_G(\omega) = \{1\}$. Denote by w_n be the prefix of ω of length n . Then by the above argument the sequence $\text{Stab}_G(w_n)$ is a nested sequence of finite index subgroups of G that are all isomorphic to G and whose intersection coincides with $\text{Stab}_G(\omega)$, which is trivial. \square

The previous corollary gives a potential way to construct essentially new examples of scale-invariant groups and is a partial motivation for this paper (see Question 4 in Section 5).

4. PROOF OF THE MAIN THEOREM.

The proof of the main theorem (Theorem 1.1) is subdivided into 4 subsections. All except two automata in the class under consideration generate either groups that act not essentially freely on ∂T_2 , or groups that have been studied before in the literature. In the first case the problem reduces to finding a nonidentity element in the rigid stabilizer of the group, while in the second case there is no need to reconstruct the structure of the group from scratch. So in both cases the analysis of the group is quite short. First we filter automata that generate groups acting not essentially freely using Mikhailova systems method and brute force methods in Subsection 4.1. Then we treat manually remaining groups whose structure has already been described (in [BGK⁺08]) in Subsection 4.2. The remaining two automata ([2193] and [2372]) generate the groups that have not been studied before and little was known about them. We completely describe the structure of these groups and prove that they act essentially freely on ∂T_2 in Subsections 4.3 and 4.4 respectively.

Our systematic search for groups that act essentially freely on ∂T_2 heavily uses results of [BGK⁺08], in conjunction with computations performed using **AutomGrp** package [MS08] developed by Y. Muntyan and the second author for **GAP** system [GAP08].

4.1. Reduction using Mikhailova system and brute force methods. We start from the list of all 194 non minimally symmetric automata (recall that this notion was introduced in Definition 11:

[1, 730, 731, 734, 739, 740, 741, 743, 744, 747, 748, 749, 750, 752, 753, 756, 766, 767, 768, 770, 771, 774, 775, 776, 777, 779, 780, 783, 802, 803, 804, 806, 807, 810, 820, 821, 824, 838, 839, 840, 842, 843, 846, 847, 848, 849, 851, 852, 855, 856, 857, 858, 860, 861, 864, 865, 866, 869, 870, 874, 875, 876, 878, 879, 882, 883, 884, 885, 887, 888, 891, 919, 920, 923, 924, 928, 929, 930, 932, 933, 936, 937, 938, 939, 941, 942, 945, 955, 956, 957, 959, 960, 963, 964, 965, 966, 968, 969, 972, 1090, 1091, 1094, 2190, 2193, 2196, 2199, 2202, 2203, 2204, 2205, 2206, 2207, 2209, 2210, 2212, 2213, 2214, 2226, 2229, 2232, 2233, 2234, 2236, 2237, 2239, 2240, 2241, 2260, 2261, 2262, 2264, 2265, 2271, 2274, 2277, 2280, 2283, 2284, 2285, 2286, 2287, 2293, 2294, 2295, 2307, 2313, 2320, 2322, 2352, 2355, 2358, 2361, 2364, 2365, 2366, 2367, 2368, 2369, 2371, 2372, 2374, 2375, 2376, 2388, 2391, 2394, 2395, 2396, 2398, 2399, 2401, 2402, 2403, 2422, 2423, 2424, 2426, 2427, 2838, 2841, 2844, 2847, 2850, 2851, 2852, 2853, 2854, 2860, 2861, 2862, 2874, 2880,

2887, 2889]

Firstly, we compute Mikhailova systems for all automata in the above list and filter out those automata, for which Mikhailova system produces a nonidentity element in the rigid stabilizer. The nontriviality of the elements listed below was checked by the program, but can be checked by hands as well. This allows us to reduce by 93 the number of automata that have to be checked. For each such automaton we list this element and its decomposition at the first level:

- | | |
|--|---|
| 741: $c^{-1}a^{-1}ba = (1, a^{-1}c^{-1}bc)$ | 945: $c^{-1}b = (1, a^{-1}b)$ |
| 744: $b^{-1}c^{-1}ba^{-1}ca = (1, a^{-1}c^{-1}ac)$ | 955: $c^{-2}ac^{-1}bca^{-1}c = (1, a^{-1}c^{-1}bc)$ |
| 749: $c^{-1}ac^{-1}ba^{-1}c = (1, a^{-1}c)$ | 956: $c^{-1}b^{-1}aba^{-1}b = (1, a^{-1}c^{-1}aba^{-1}c)$ |
| 753: $b^{-1}aba^{-1}bc^{-1} = (1, a^{-1}bcb^{-1})$ | 957: $c^{-2}aba^{-1}c^{-1}ac^{-1}ac = (1, a^{-1}b)$ |
| 776: $a^{-1}ba^{-1}c = (1, b^{-1}c)$ | 959: $c^{-1}b^{-1}aba^{-1}b = (1, a^{-1}c^{-1}bc)$ |
| 777: $c^{-1}b^{-1}a^2 = (1, a^{-1}b^{-1}ac)$ | 960: $aba^{-1}ba^{-2}c^{-1}aba^{-1} = (1, a^{-1}b)$ |
| 779: $c^{-1}ab^{-1}cba^{-1} = (1, a^{-1}bc^{-1}acb^{-1})$ | 963: $c^{-1}aba^{-1} = (1, a^{-1}cbc^{-1})$ |
| 840: $b^{-1}a^{-1}ca = (1, b^{-1}c^{-1}bc)$ | 965: $c^{-1}b = (1, a^{-1}c)$ |
| 843: $c^{-1}a^{-1}ba = (1, a^{-1}c^{-1}ac)$ | 969: $c^{-1}b = (1, a^{-1}c)$ |
| 849: $c^{-1} = (1, a^{-1})$ | 2199: $b^{-1}a = (1, b^{-1}c)$ |
| 852: $c^{-1} = (1, a^{-1})$ | 2202: $cb^{-1}c^{-1}b = (1, ab^{-1}a^{-1}b)$ |
| 856: $c^{-2}ac^{-1}bca^{-1}c = (1, a^{-1}b^{-1}cb)$ | 2203: $c^{-2}ab = (1, a^{-2}cb)$ |
| 857: $c^{-2}ac^{-1}ac^{-1}aba^{-1}c = (1, a^{-1}c)$ | 2204: $c^{-1}ab^{-1}c = (1, a^{-1}c)$ |
| 858: $c^{-1}b^{-1}aba^{-1}b = (1, a^{-1}b^{-1}aca^{-1}b)$ | 2207: $a^{-1}b = (1, b^{-1}c)$ |
| 860: $c^{-1}aba^{-1} = (1, a^{-1}bcb^{-1})$ | 2209: $c^{-1}aca^{-1} = (1, a^{-1}bab^{-1})$ |
| 861: $c^{-1}b^{-1}aba^{-1}b = (1, a^{-1}c)$ | 2210: $c^{-1}b^{-1}cb = (1, a^{-1}c^{-1}ac)$ |
| 864: $c^{-1}b^{-1}aba^{-1}b = (1, a^{-1}b^{-1}cb)$ | 2213: $c^{-1}b^{-1}cb = (1, a^{-1}c^{-1}ac)$ |
| 866: $c^{-1}ac^{-1}a^{-1}cba^{-1}ca^{-1}c = (1, b^{-1}c)$ | 2234: $c^{-1}b^{-1}ac^{-1}a^2 = (1, a^{-1}c^{-1}b^2)$ |
| 869: $c^{-1}ac^{-1}a^{-1}cb = (1, a^{-1}b^{-1}ac)$ | 2236: $a^{-1}b = (1, a^{-1}c)$ |
| 874: $c^{-1}b = (1, a^{-1}c)$ | 2239: $ca^{-2}cba^{-1} = (1, c^{-1}abc^{-1})$ |
| 875: $c^{-1}ac^{-1}ac^{-1}b = (1, a^{-1}c)$ | 2261: $c^{-1}a^{-1}ca = (1, a^{-1}b^{-1}ab)$ |
| 876: $c^{-1}b = (1, a^{-1}c)$ | 2271: $b^{-1}a = (1, a^{-1}c)$ |
| 878: $c^{-1}b = (1, a^{-1}c)$ | 2274: $c^{-3}bc^2b^{-3}c^3 = (1, a^{-2}b^2)$ |
| 879: $c^{-1}b = (1, a^{-1}c)$ | 2280: $b^{-2}a^2b^{-1}a^{-1}b^2 = (1, b^{-1}c)$ |
| 882: $c^{-1}b = (1, a^{-1}c)$ | 2283: $c^{-2}bac^{-2}bcb^{-1}cb^{-2}c^2 = (1, a^{-1}c)$ |
| 883: $c^{-1}b^{-1}ac^{-1}bca^{-1}c = (1, a^{-1}c^{-1}ab^{-1}cb)$ | 2284: $c^{-1}bca^{-1} = (1, bc^{-1})$ |
| 885: $c^{-1}b^{-1}aba^{-1}c = (1, a^{-1}c^{-1}ac)$ | 2285: $c^{-1}ac^{-1}b = (1, a^{-1}c)$ |
| 887: $c^{-1}ab^{-1}a^{-1}cb = (1, a^{-1}bc^{-1}b^{-1}ac)$ | 2287: $c^{-1}b^{-1}c^2a^{-1}c = (1, a^{-1}c^{-1}b^2)$ |
| 888: $c^{-1}ab^{-1}a = (1, a^{-1}b)$ | 2293: $b^{-1}c^2a^{-1} = (1, c^{-1}b^2c^{-1})$ |
| 920: $b^{-1}ab^{-1}cba^{-1}ba^{-1}ba^{-1} = (1, b^{-1}c)$ | 2295: $c^{-1}ab^{-1}c = (1, a^{-1}c)$ |
| 923: $b^{-1}ab^{-1}c^{-1}ba^{-1}b^2 = (1, a^{-1}c^{-1}ab)$ | 2307: $c^{-1}bc^{-1}a = (1, a^{-1}c)$ |
| 929: $c^{-1}a^{-1}ca^{-1}c = (1, a^{-1}c)$ | 2322: $ba^{-1} = (1, bc^{-1})$ |
| 933: $c^{-1}a^2 = (1, a^{-1}c)$ | 2355: $b^{-1}a^{-1}cb^{-1}cb = (1, a^{-1}b^{-1}ca)$ |
| 937: $c^{-1}b = (1, a^{-1}b)$ | 2361: $b^{-1}a = (1, b^{-1}c)$ |
| 938: $c^{-1}b = (1, a^{-1}b)$ | 2364: $c^{-1}ac^{-1}b = (1, a^{-1}bc^{-1}b)$ |
| 939: $c^{-1}bc^{-1}ac^{-1}a = (1, a^{-1}b)$ | 2365: $ac^{-1}ac^{-1}b^{-1}c^2a^{-1} = (1, b^{-1}c)$ |
| 941: $c^{-1}b = (1, a^{-1}b)$ | 2366: $ba^{-1} = (1, ac^{-1})$ |
| 942: $aba^{-1}ba^{-2}c^{-1}b = (1, a^{-1}b)$ | 2367: $aca^{-1}cb^{-1}a^{-1} = (1, cab^{-1}c^{-1})$ |

2369: $a^{-1}b = (1, b^{-1}c)$	2403: $ba^{-1} = (1, bc^{-1})$
2371: $ac^{-2}b = (1, bc^{-1})$	2423: $c^{-1}bc^{-1}a = (1, a^{-1}b)$
2375: $c^{-1}b^{-1}ca = (1, a^{-1}b)$	2427: $ab^{-1} = (1, bc^{-1})$
2395: $b^{-1}ca^{-1}c^{-1}b^2 = (1, a^{-2}bc)$	2841: $b^{-1}a^{-1}ba^{-1} = (1, a^{-1}b^{-1}ab^{-1})$
2396: $c^{-1}bc^{-1}a = (1, a^{-1}bc^{-1}b)$	2847: $b^{-1}a = (1, b^{-1})$
2398: $a^{-1}b = (1, a^{-1}c)$	2850: $b^{-1}a^2b^{-1}ab = (1, a^{-1}b^2)$
2399: $a^{-1}b = (1, b^{-1}c)$	2851: $a^{-3}b = (1, a^{-2}b)$
2401: $c^{-1}bca^{-1} = (1, a^{-1}b)$	2852: $ab^{-1} = (1, a^{-1})$
2402: $c^{-2}ba = (1, a^{-2}b^2)$	

For the remaining 101 automata we applied a brute force in an attempt to find nonidentity elements in the rigid stabilizer of the first level up to length 5 using the function `FindGroupElement` of `AutomGrp` package. This allows to eliminate the following groups.

739: $bc = (ba, 1)$	2205: $(ab)^2 = (1, (cb)^2)$
740: $bc^{-1} = (ba^{-1}, 1)$	2206: $ab = (1, ac)$
743: $bc = (ba, 1)$	2212: $abc^{-2} = (1, c^2a^{-2})$
747: $bc = (ba, 1)$	2214: $ab = (ca, 1)$
748: $bc = (ca, 1)$	2229: $ab = (cb, 1)$
750: $bc^{-1} = (ca^{-1}, 1)$	2233: $bc b^{-1}c = (1, bab^{-1}a)$
752: $bc = (ca, 1)$	2237: $ab^{-1} = (bc^{-1}, 1)$
756: $bc = (ca, 1)$	2241: $ab = (cb, 1)$
775: $bcb c = (1, baba)$	2262: $ab^{-1} = (1, ac^{-1})$
780: $(bc^{-1})^2 = ((ca^{-1})^2, 1)$	2265: $ab^{-1} = (1, bc^{-1})$
783: $(bc)^2 = (1, (ba)^2)$	2286: $ab^{-1}ab^{-1} = (1, ca^{-1}ca^{-1})$
838: $abac = ((ab)^2, 1)$	2352: $ab^{-1} = (ca^{-1}, 1)$
842: $abac = (1, (ba)^2)$	2368: $ab = (1, ac)$
847: $c = (1, a)$	2376: $ab = (ca, 1)$
848: $c = (1, a)$	2391: $ab = (cb, 1)$
851: $c = (1, a)$	2424: $ab^{-1} = (1, ac^{-1})$
855: $c = (1, a)$	2838: $ab^{-1} = (a^{-1}, 1)$
964: $bc = (1, ca)$	2853: $(ab)^2 = (1, b^2)$
966: $bc^{-1} = (1, ca^{-1})$	2854: $ab = (1, a)$
968: $bc = (1, ca)$	2860: $ab = (a^2, 1)$
972: $bc = (1, ca)$	2862: $ab = (a, 1)$
2190: $ab^{-1} = (ca^{-1}, 1)$	2889: $ab = (b, 1)$

The above reduction leaves the following 57 candidates for automata that generate groups acting essentially freely:

[1, 730, 731, 734, 766, 767, 768, 770, 771, 774, 802, 803, 804, 806, 807, 810, 820, 821, 824, 839, 846, 865, 870, 884, 891, 919, 924, 928, 930, 932, 936, 1090, 1091, 1094, 2193, 2196, 2226, 2232, 2240, 2260, 2264, 2277, 2294, 2313, 2320, 2358, 2372, 2374, 2388, 2394, 2422, 2426, 2844, 2861, 2874, 2880, 2887]

In the next three subsections, we investigate these cases separately.

4.2. Investigation of easy cases. The format of this subsection is as follows. Some automata listed in the end of previous subsection generate isomorphic groups, for which the proof of essential freeness/non-freeness is identical. We then unite such groups into one case. Other automata are treated separately. We start each

case by listing the numbers of automata from the list at the the end of previous subsection treated in there (these numbers are given in bold font). Within each case we mean by G the group generated by an automaton under consideration.

1. This automaton generates a trivial group which by definition acts essentially freely on ∂T_2 .

730,734,766,770,774,2232,2264,2844,2880. All automata in this list generate the Klein group of order 4 isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Straightforward check reveals that no nonidentity element of any of these groups belongs to $\text{Rist}_G(1)$. Thus by Proposition 3.4 these groups act essentially freely on ∂T_2 .

731,767,768,804,1091,2861,2887. All automata in this family generate groups isomorphic to \mathbb{Z} . We will prove now that if an automaton generates $G \cong \mathbb{Z}$, then the action of G on ∂T_2 is essentially free. Suppose not, then by Proposition 3.4 and spherical transitivity of a group (as \mathbb{Z} is infinite acting on the binary tree) there must be a nonidentity element $g = (1, g|_1)$ in $\text{Rist}_G(1)$. Since G is nontrivial, by self-similarity there must be an element $h = (h|_0, h|_1)\sigma \in G$ that acts nontrivially on the first level. Conjugating g by h yields

$$g^h = (h_1^{-1}, h_0^{-1})\sigma \cdot (1, g|_1) \cdot (h|_0, h|_1)\sigma = (g|_1^{h|_1}, 1).$$

Since both g and g^h are nonidentity elements of $G \cong \mathbb{Z}$, there must be $n, m \in \mathbb{Z} - \{0\}$ such that $g^n = (g^h)^m$, which implies

$$(1, g|_1^n) = ((g|_1^{h|_1})^m, 1).$$

This is a contradiction because $g|_1 \in G$ has an infinite order as each nonidentity element of G . Thus G acts essentially freely on ∂T_2 .

771. The wreath recursion for G_{771} is $a = (c, 1)\sigma, c = (a, a)$ and the group G it generates is isomorphic to \mathbb{Z}^2 freely generated by a and c . Each element of a stabilizer of the first level can be written as $a^{2^n}c^m$ for some $n, m \in \mathbb{Z}$. Since

$$a^{2^n}c^m = (c^n a^m, c^n a^m),$$

the only time this element belongs to $\text{Rist}_G(1)$ is when $n = m = 0$, i.e. $a^{2^n}c^m = 1$. Thus, $\text{Rist}_G(1)$ is trivial and G acts essentially freely on ∂T_2 by Proposition 3.4.

802,806,810,2196,2260. All automata in this list generate an abelian group of order 8 isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Straightforward check reveals that no nonidentity element of any of these groups belongs to $\text{Rist}_G(1)$. Thus by Proposition 3.4 these groups act essentially freely on ∂T_2 .

803. The wreath recursion for G_{803} is $a = (b, a)\sigma, b = (c, c), c = (a, a)$ and the group G it generates is isomorphic to \mathbb{Z}^2 freely generated by a and b (where $c = a^{-2}b^{-1}$). Each element of a stabilizer of the first level can be written as $a^{2^n}b^m$ for some $n, m \in \mathbb{Z}$. Since

$$a^{2^n}b^m = (a^n b^n c^m, a^n b^n c^m)$$

and the sections at the vertices of the first level are equal, the only time this element belongs to $\text{Rist}_G(1)$ is when these sections are trivial, i.e. $m = n = 0$ and,

hence, $a^{2n}b^m = 1$. Thus, $\text{Rist}_G(1)$ is trivial and G acts essentially freely on ∂T_2 by Proposition 3.4.

807. The wreath recursion for G_{807} is $a = (c, b)\sigma, b = (c, c), c = (a, a)$ and the group G it generates is isomorphic to \mathbb{Z}^2 freely generated by a and c (where $b = a^{-2}c^{-2}$). Each element of a stabilizer of the first level can be written as $a^{2n}c^m$ for some $n, m \in \mathbb{Z}$. Since

$$a^{2n}c^m = (c^n b^n a^m, c^n b^n a^m)$$

and the sections at the vertices of the first level are equal, the only time this element belongs to $\text{Rist}_G(1)$ is when these sections are trivial, i.e. $m = n = 0$ and, hence, $a^{2n}b^m = 1$. Thus, $\text{Rist}_G(1)$ is trivial and G acts essentially freely on ∂T_2 by Proposition 3.4.

820,824,865,919,928,932,936,2226,2358,2394,2422,2874. All automata in this family generate D_∞ . We will prove that if an automaton generates $G \cong D_\infty$, then the action of G on ∂T_2 is essentially free by the same method we used for automata generating \mathbb{Z} . Suppose not, then by Proposition 3.4 and spherical transitivity of a group (as D_∞ is infinite acting on the binary tree) there must be a nonidentity element $g = (1, g|_1)$ in $\text{Rist}_G(1)$. Since G is nontrivial, by self-similarity there must be an element $h = (h|_0, h|_1)\sigma \in G$ that acts nontrivially on the first level. Conjugating g by h yields

$$g^h = (h_1^{-1}, h_0^{-1})\sigma \cdot (1, g|_1) \cdot (h|_0, h|_1)\sigma = (g|_1^{h|_1}, 1).$$

Both g and g^h are different nonidentity elements of $G \cong D_\infty$ that commute. All centralizers of nonidentity elements in D_∞ are cyclic either of order 2 or infinite. Since we have already two nonidentity elements in the $C_G(g)$, this subgroup has to be isomorphic to \mathbb{Z} . Hence, there must be $n, m \in \mathbb{Z} - \{0\}$ such that $g^n = (g^h)^m$, which implies

$$(1, g|_1^m) = ((g|_1^{h|_1})^m, 1).$$

This is a contradiction because $g|_1 \in G$ has an infinite order as g has an infinite order. Thus G acts essentially freely on ∂T_2 .

821. The group G_{821} generated by this automaton is isomorphic to the lamplighter group $\mathcal{L} \cong (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ (see [GNS00]) and has the following presentation:

$$(6) \quad G \cong \langle a, b \mid [(b^{-1}a), (b^{-1}a)^{b^i}] = (b^{-1}a)^2 = 1, i \geq 1 \rangle,$$

that can be obtained from the standard presentation $\langle x, y \mid [x, x^{y^i}] = x^2 = 1 \rangle$ of \mathcal{L} by Tietze transformations.

The Mikhailova system for this group is

$$\begin{aligned} b^{-1}aba^{-1}b &= (a, b) \\ b &= (b, a). \end{aligned}$$

Therefore, by Proposition 3.5 it is enough to prove that the map $\phi: F_2 \rightarrow F_2$ defined by

$$\begin{aligned} \phi(a) &= b, \\ \phi(b) &= a, \end{aligned}$$

induces an automorphism of \mathcal{L} .

To prove that the relators in presentation (6) are mapped by ϕ to the identity element we first show by induction that

$$(b^{-1}a)^{b^i} = (b^{-1}a)^{a^i}$$

for all $i \geq 0$. For $i = 0$ there is nothing to prove. The induction step is proved as follows:

$$(b^{-1}a)^{b^{i+1}} = \left((b^{-1}a)^{b^i}\right)^b = \left((b^{-1}a)^{a^i}\right)^b = \left((b^{-1}a)^{a^{i+1}}\right)^{a^{-1}b} = (b^{-1}a)^{a^{i+1}}.$$

Therefore, for the relators in presentation (6) we have:

$$\begin{aligned} \phi([(b^{-1}a), (b^{-1}a)^{b^i}]) &= [(a^{-1}b), (a^{-1}b)^{a^i}] = [(a^{-1}b), (a^{-1}b)^{b^i}] = 1, \\ \phi((b^{-1}a)^2) &= (a^{-1}b)^2 = 1 \end{aligned}$$

Thus ϕ induces a surjective endomorphism of G . Since \mathcal{L} is residually finite, it has a Hopf property, so ϕ must be an isomorphism.

839. The wreath recursion of this automaton is $a = (b, a)\sigma, b = (a, b), c = (b, a)$. Since $c = aba^{-1}$ we get $G = \langle a, b \rangle \cong \mathcal{L}$. The proof that the action of G on ∂T_2 is essentially free is now identical to the one for the automaton 821.

846. This is an automaton called Bellaterra (or, sometimes, baby-Aleshin) automaton generating a free product of three groups of order 2: $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ [Nek05, BGK⁺08]. The automaton \mathcal{A}_{846} is bireversible, so by Proposition 3.6 the group it generates acts essentially freely on ∂T_2 .

870. This automaton generates a group isomorphic to the Baumslag-Solitar group $BS(1, 3)$ (see [BGK⁺08]). It is proved (using Proposition 3.5) in Example 5.5 in [Gri11] that this group acts essentially freely on ∂T_2 .

884. The wreath recursion for G_{884} is $a = (b, a)\sigma, b = (c, c), c = (b, a)$. The Mikhailova system for this group is

$$\begin{aligned} u &:= c^{-1}a^2b^{-1}c^{-1}aba^{-1}c &= (a, b) \\ c & &= (b, a) \\ b & &= (c, c) \end{aligned}$$

Since $[b, c^{-1}a] = 1$ in G , but $[a, c^{-1}b] \neq 1$ in G we get that the rigid stabilizer of the first level contains a nonidentity element $[c, b^{-1}u] = ([b, c^{-1}a], [a, c^{-1}b]) = (1, [a, c^{-1}b])$. Thus the action on the boundary of the tree is not essentially free.

891. The wreath recursion for G_{891} is $a = (c, c)\sigma, b = (c, c), c = (b, a)$. It is shown in [BGK⁺08] that the group generated by this automaton is isomorphic to $\mathcal{L} \rtimes (Z/2\mathbb{Z}) = ((\mathbb{Z}/2\mathbb{Z}) \wr Z) \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $\mathcal{L} \cong L := \langle x = ca, y = bc \rangle$, and $\mathbb{Z}/2\mathbb{Z} = \langle c \rangle$ acts on L by inversion of x and y . It follows, that G has the following presentation with respect to the generating set $\{x, y, c\}$:

$$(7) \quad G \cong \langle x, y, c \mid [yx, (yx)^{y^i}] = (yx)^2 = 1, x^c = x^{-1}, y^c = y^{-1}, c^2 = 1, i \geq 1 \rangle$$

This presentation by Tietze transformations (using expression of x and y in terms of a, b and c) can be converted to the following presentation:

$$G \cong \langle a, b, c \mid [bc^2a, (bc^2a)^{(bc)^i}] = (bc^2a)^2 = 1, \\ c^{-1}cac = a^{-1}c^{-1}, i \geq 1, c^{-1}bcc = c^{-1}b^{-1}, c^2 = 1 \rangle$$

that simplifies to

$$(8) \quad G \cong \langle a, b, c \mid [ba, (ba)^{(bc)^i}] = (ba)^2 = a^2 = b^2 = c^2 = 1, i \geq 1 \rangle$$

The Mikhailova system for this automaton is

$$\begin{aligned} b^{-1}aca^{-1}b &= (a, b), \\ c &= (b, a), \\ b &= (c, c). \end{aligned}$$

Therefore, by Proposition 3.5 it is enough to prove that the map $\phi: F_3 \rightarrow F_3$ defined by

$$\begin{aligned} \phi(a) &= b, \\ \phi(b) &= a, \\ \phi(c) &= c \end{aligned}$$

induces an automorphism of G .

To prove that the relators in presentation (8) are mapped by ϕ to the identity element we first show by induction that

$$(ab)^{(ac)^i} = (ab)^{(bc)^i}$$

for all $i \geq 0$. For $i = 0$ there is nothing to prove. The induction step is proved as follows:

$$(ab)^{(ac)^{i+1}} = \left((ab)^{(ac)^i} \right)^{ac} = \left((ab)^{(bc)^i} \right)^{ac} = \left((ab)^{(bc)^{i+1}} \right)^{(ab)^{bc}} = (ab)^{(bc)^{i+1}}.$$

Therefore, for the relators in presentation (6) we have:

$$\begin{aligned} \phi([ba, (ba)^{(bc)^i}]) &= [ab, (ab)^{(ac)^i}] = [(ab), (ab)^{(bc)^i}] = 1, \\ \phi((ba)^2) &= (ab)^2 = 1, \\ \phi(a^2) &= b^2 = 1, \quad \phi(b^2) = a^2 = 1, \quad \phi(c^2) = c^2 = 1. \end{aligned}$$

Thus ϕ induces a surjective endomorphism of G . Since \mathcal{L} is residually finite, it has a Hopf property, so ϕ must be an isomorphism.

924. The group generated by this automaton is isomorphic to $BS(1, 3)$ [BS06] and has the following presentation:

$$G \cong \langle a, b, c \mid (ac^{-1})^a(ac^{-1})^{-3} = ba^{-1}ca^{-1} = 1 \rangle,$$

that can be obtained from the standard presentation of $BS(1, 3)$ by Tietze transformations.

The Mikhailova system for this automaton is

$$\begin{aligned} c^{-1}aca^{-1}c &= (a, a^{-1}bcb^{-1}a), \\ c^{-1}a^2 &= (b, a^{-1}bc), \\ c &= (c, a), \\ c^{-1}ab^{-1}a &= (1, a^{-1}ba^{-1}c), \\ c^{-1}ac^{-1}b^{-1}ab &= (1, a^{-1}bc^{-1}a^{-1}cb), \end{aligned}$$

where $a^{-1}ba^{-1}c = a^{-1}bc^{-1}a^{-1}cb = 1$ in G .

Therefore, by Proposition 3.5 it is enough to prove that the map $\phi: F_3 \rightarrow F_3$ defined by

$$\begin{aligned}\phi(a) &= a^{-1}bcb^{-1}a, \\ \phi(b) &= a^{-1}bc, \\ \phi(c) &= a\end{aligned}$$

induces an automorphism of G . Now we use `automgrp` package to verify that the relators of G are mapped by ϕ to the identity element in G :

```
gap> A:=a^-1*b*c*b^-1*a;
a^-1*b*c*b^-1*a
gap> B:=a^-1*b*c;
a^-1*b*c
gap> C:=a;
a
gap> IsOne((A*C^-1)^A*(A*C^-1)^-3);
true
gap> IsOne(B*A^-1*C*A^-1);
true
```

Thus ϕ induces a surjective endomorphism of G . Since $BS(1, 3)$ is residually finite, it has a Hopf property, so ϕ must be an isomorphism.

930. The wreath recursion for G_{930} is $a = (c, a)\sigma, b = (b, b), c = (c, a)$. Since b is the identity state, the group generated by this automaton coincides with the lamplighter group generated by automaton 821, which acts essentially freely on ∂T_2 .

1090, 1094. Both these automata generate a group of order 2, whose nonidentity element does not belong to the rigid stabilizer as it has to act nontrivially on the vertices of the first level. Thus by Proposition 3.4 these groups act essentially freely on ∂T_2 .

2240. This is an Aleshin automaton (originally constructed in [Ale83]) generating a free group F_3 of rank 3 [VV07]. The automaton itself is bireversible, so by Proposition 3.6 the group it generates acts essentially freely on ∂T_2 .

2277. The wreath recursion for G_{2277} is $a = (c, c)\sigma, b = (a, a)\sigma, c = (b, a)$ and the group G it generates is isomorphic to $\mathbb{Z}^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ as shown in [BGK⁺08]. More precisely, elements $x = bc$ and $y = ba$ freely generate \mathbb{Z}^2 and

$$G \cong \langle x, y \rangle \rtimes \langle b \rangle,$$

where b is an element of order 2 acting nontrivially on the first level and acting on $\langle x, y \rangle$ by conjugation inverting each element.

Consider first elements in $\langle x, y \rangle$. We have the following wreath recursion for x and y :

$$\begin{aligned}x &= (1, y^{-1})\sigma, \\ y &= (xy^{-1}, xy^{-1}).\end{aligned}$$

Each element of a stabilizer of the first level of $\langle x, y \rangle$ can be written as $x^{2n}y^m$ for some $n, m \in \mathbb{Z}$. Since

$$x^{2n}y^m = (x^m y^{-m-n}, x^m y^{-m-n})$$

and the sections at the vertices of the first level are equal, the only time this element belongs to $\text{Rist}_G(1)$ is when these sections are trivial, i.e. $x^{2n}y^m = 1$.

Each element in $\text{Stab}_G(1)$ which is not in $\langle x, y \rangle$ can be written as

$$x^{2n+1}y^mb = (x^my^{-m-n}a, x^my^{-m-n-1}a).$$

Both of the sections of the latter element are nontrivial since $a \notin \langle x, y \rangle$. Therefore, this element cannot belong to $\text{Rist}_G(1)$.

Thus, $\text{Rist}_G(1)$ is trivial and G acts essentially freely on ∂T_2 by Proposition 3.4.

2294. The group generated by this automaton is isomorphic to $BS(1, -3)$ with the following presentation with respect to generators a, b, c (see [BGK⁺08]):

$$G \cong \langle a, b, c \mid a^{-1}(a^{-1}c)a(a^{-1}c)^3 = ca^{-1}cb^{-1} = 1 \rangle,$$

that can be obtained from the standard presentation of $BS(1, -3)$ by Tietze transformations.

The Mikhailova system for this automaton is

$$\begin{aligned} c^{-1}aca^{-1}cba^{-1}ca^{-2}c &= (a, a^{-1}cbc^{-1}a^2c^{-1}ab^{-1}c^{-1}a), \\ c &= (b, a), \\ c^{-1}a^2ba^{-1}ca^{-2}c &= (c, a^{-1}cbac^{-1}ab^{-1}c^{-1}a), \\ c^{-1}ac^{-1}b &= (1, a^{-1}cb^{-1}c), \\ c^{-2}a^2c^{-1}ab^{-1}c &= (1, a^{-2}cba^{-1}c), \end{aligned}$$

where $a^{-1}cb^{-1}c = a^{-2}cba^{-1}c = 1$ in G .

Now by Proposition 3.5 it is enough to prove that the map $\phi: F_3 \rightarrow F_3$ defined by

$$\begin{aligned} \phi(a) &= a^{-1}cbc^{-1}a^2c^{-1}ab^{-1}c^{-1}a, \\ \phi(b) &= a, \\ \phi(c) &= a^{-1}cbac^{-1}ab^{-1}c^{-1}a \end{aligned}$$

induces an automorphism of G . We use `automgrp` package to verify that the relators of G are mapped to the identity element in G :

```
gap> G:=AutomatonGroup("a=(b,c)(1,2),b=(c,a)(1,2),c=(b,a)");
< a, b, c >
gap> A:=a^-1*c*b*c^-1*a^2*c^-1*a*b^-1*c^-1*a;
a^-1*c*b*c^-1*a^2*c^-1*a*b^-1*c^-1*a
gap> B:=a;
a
gap> C:=a^-1*c*b*a*c^-1*a*b^-1*c^-1*a;
a^-1*c*b*a*c^-1*a*b^-1*c^-1*a
gap> IsOne(A^-2*C*A*(A^-1*C)^3);
true
gap> IsOne(C*A^-1*C*B^-1);
true
```

Thus, ϕ induces a surjective endomorphism of G . Since $BS(1, -3)$ is residually finite, it has a Hopf property, so ϕ must be an isomorphism.

2313. The wreath recursion for G_{2313} is $a = (c, c)\sigma, b = (b, b)\sigma, c = (b, a)$ and the group G it generates is isomorphic to $\mathbb{Z}^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ as shown in [BGK⁺08]. Elements

$x = ab$ and $y = cb$ freely generate \mathbb{Z}^2 and

$$G \cong \langle x, y \rangle \rtimes \langle b \rangle,$$

where b is an element of order 2 acting nontrivially on the first level and acting on $\langle x, y \rangle$ by conjugation inverting each element.

Consider first elements in $\langle x, y \rangle$. We have the following wreath recursion for x and y :

$$\begin{aligned} x &= (y, y), \\ y &= (1, x)\sigma. \end{aligned}$$

This recursion coincides with the definition of an automaton 771, which generates a group acting essentially freely on ∂T_2 . On the other hand, G can be defined by wreath recursion $x = (y, y), y = (1, x)\sigma, b = (b, b)\sigma$. Since b has order 2, each element g in the complement of $\langle x, y \rangle$ in G can be written as wb , where $w \in \langle x, y \rangle$. Both sections of g will be words in x, y and b containing exactly one b (since $b = (b, b)\sigma$). Thus, these sections cannot be trivial since $b \notin \langle x, y \rangle$.

Thus, the group G also acts essentially freely on ∂T_2 .

2320. The group generated by this automaton is also isomorphic to $BS(1, -3)$ with the following presentation with respect to generators a, b, c :

$$G \cong \langle a, b, c \mid a(c^{-1}a)a^{-1}(c^{-1}a)^3 = ca^{-1}cb^{-1} = 1 \rangle,$$

that can be obtained from the standard presentation of $BS(1, -3)$ by Tietze transformations.

Indeed, these relations do hold in G :

```
gap> G:=AutomatonGroup("a=(a,c)(1,2),b=(c,b)(1,2),c=(b,a)");
< a, b, c >
gap> IsOne((c^-1*a)^(a^-1)*(c^-1*a)^3);
true
gap> IsOne(c*a^-1*c*b^-1);
true
```

And since both a and $c^{-1}a$ are of infinite order:

```
gap> Order(a);
infinity
gap> Order(c^-1*a);
infinity
```

we have an isomorphism $G \cong BS(1, -3)$.

The Mikhailova system for this automaton is

$$\begin{aligned} aca^{-1} &= (a, cbc^{-1}), \\ c &= (b, a), \\ bc^{-1}bc^{-1}aca^{-1} &= (c, ca^{-1}cbc^{-1}), \\ c^{-1}ac^{-1}b &= (1, a^{-1}cb^{-1}c), \\ a^{-1}ca^{-1}bc^{-1}bc^{-1}aca^{-1} &= (1, a^{-1}ba^{-1}cbc^{-1}), \end{aligned}$$

where $a^{-1}cb^{-1}c = a^{-1}ba^{-1}cbc^{-1} = 1$ in G .

Now by Proposition 3.5 it is enough to prove that the map $\phi: F_3 \rightarrow F_3$ defined by

$$\begin{aligned}\phi(a) &= cbc^{-1}, \\ \phi(b) &= a, \\ \phi(c) &= ca^{-1}cbc^{-1}\end{aligned}$$

induces an automorphism of G . We use **automgrp** package to verify that the relators of G are mapped to the identity element in G :

```
gap> A:=c*b*c^-1;
c*b*c^-1
gap> B:=a;
a
gap> C:=c*a^-1*c*b*c^-1;
c*a^-1*c*b*c^-1
gap> IsOne((C^-1*A)^(A^-1)*(C^-1*A)^3);
true
gap> IsOne(C*A^-1*C*B^-1);
true
```

Thus, ϕ induces an surjective endomorphism of G . Since $BS(1, -3)$ is residually finite, it has a Hopf property, so ϕ must be an isomorphism.

2374. The wreath recursion for G_{2374} is $a = (a, c)\sigma, b = (c, a)\sigma, c = (c, a)$. Since $b = cac^{-1}$, the group generated by this automaton coincides as a subgroup of $\text{Aut}(T_2)$ with the lamplighter group generated by automaton 930, which acts essentially freely on ∂T_2 .

2388. The wreath recursion for G_{2388} is $a = (c, a)\sigma, b = (b, b)\sigma, c = (c, a)$. Since $b = \sigma = c^{-1}a$, the group generated by this automaton coincides as a subgroup of $\text{Aut}(T_2)$ with the lamplighter group generated by automaton 821, which act essentially freely on ∂T_2 .

2426. The wreath recursion for G_{2426} is $a = (b, b)\sigma, b = (c, c)\sigma, c = (c, a)$ and the group G it generates is isomorphic to $\mathbb{Z}^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ as shown in [BGK⁺08]. The proof of essential freeness is identical to the one for the automaton 2277. The elements $x = ba$ and $y = bc$ freely generate \mathbb{Z}^2 and

$$G \cong \langle x, y \rangle \rtimes \langle b \rangle,$$

where b is an element of order 2 acting nontrivially on the first level.

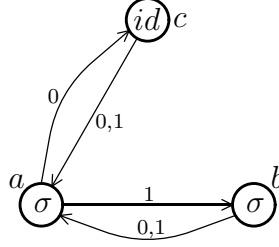
Consider first elements in $\langle x, y \rangle$. We have the following wreath recursion for x and y :

$$\begin{aligned}x &= (y^{-1}, y^{-1}), \\ y &= (y^{-1}x, 1)\sigma.\end{aligned}$$

Each element of the stabilizer of the first level of $\langle x, y \rangle$ can be written as $x^n y^{2m}$ for some $n, m \in \mathbb{Z}$. Since

$$x^n y^{2m} = (x^m y^{-m-n}, x^m y^{-m-n})$$

and the sections at the vertices of the first level are equal, the only time this element belongs to $\text{Rist}_G(1)$ is when these sections are trivial, i.e. $x^n y^{2m} = 1$.

FIGURE 1. Automaton \mathcal{A}_{2193} generating G_{2193}

On the other hand, both sections of each element in the complement of $\langle x, y \rangle$ in G will be words in x, y and c containing exactly one c . Thus, these sections cannot be trivial since $c \notin \langle x, y \rangle$.

Thus, $\text{Rist}_G(1)$ is trivial and G acts essentially freely on ∂T_2 by Proposition 3.4.

The only two remaining automata to consider are automata \mathcal{A}_{2193} and \mathcal{A}_{2372} . We devote the next two subsections to the complete analysis of these two special cases.

4.3. Automaton 2193. Throughout this section we denote by G the group G_{2193} generated by automaton \mathcal{A}_{2193} and defined by the following wreath recursion: $a = (c, b)\sigma$, $b = (a, a)\sigma$, $c = (a, a)$. The automaton \mathcal{A}_{2193} itself is depicted in Figure 1. Our goal in this subsection is to prove the following structure theorem for G , that will allow us to prove that the action of G on ∂T_2 is essentially free in Corollary 4.12.

Theorem 4.1. *The group $G = \langle a, b, c \rangle = \langle a^2, b^{-1}c, b^{-1}a, ac^{-1}a \rangle$ is solvable of derived length 3 and has the following structure:*

$$G \cong \mathcal{L}_{2,2} \rtimes (\mathbb{Z}/2\mathbb{Z}) = ((\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where the isomorphism is induced by sending the first two generators $a^2, b^{-1}c$ of G to generators from the second generating set of the base group $(\mathbb{Z}/2\mathbb{Z})^2$ in $\mathcal{L}_{2,2}$, the generator $b^{-1}a$ to the generator of \mathbb{Z} in $\mathcal{L}_{2,2}$, and the generator $ac^{-1}a$ of G to the generator of $\mathbb{Z}/2\mathbb{Z}$ in $\mathcal{L}_{2,2} \rtimes (\mathbb{Z}/2\mathbb{Z})$ acting on $\mathcal{L}_{2,2}$ according to the following rules:

$$(9) \quad \begin{aligned} (b^{-1}c)^t &= (b^{-1}a)^{-1}(b^{-1}c)a^2y^{-1}(b^{-1}c)^{-1}(b^{-1}a)a^2(b^{-1}c)^{-1}(b^{-1}a), \\ (b^{-1}a)^t &= (b^{-1}a)^{-1}(b^{-1}c)a^2y^{-1}(b^{-1}c)^{-1}(b^{-1}a), \\ (a^2)^t &= (b^{-1}a)^{-1}(b^{-1}c)a^2(b^{-1}c)^{-1}(b^{-1}a). \end{aligned}$$

Moreover, the group G has the following presentation:

$$(10) \quad G \cong \langle a, b, c \mid a^4 = (b^{-1}c)^2 = 1, \\ \left[a^2, (a^2)^{(b^{-1}a)^i} \right] = \left[a^2, (b^{-1}c)^{(b^{-1}a)^i} \right] = \left[b^{-1}c, (b^{-1}c)^{(b^{-1}a)^i} \right] = 1, \ i \geq 1, \\ (ba^2)^2 = (ca^2)^2 = 1 \rangle$$

We begin from the introduction of necessary notation and technical lemmas. It is shown in [BGK⁺08] that a group $L = \langle x = a^{-1}c, y = b^{-1}a \rangle$ is isomorphic to the lamplighter group \mathcal{L} , and this group acts on X^* in a self-similar way via the following wreath recursion:

$$(11) \quad \begin{aligned} x &= (y, x^{-1})\sigma, \\ y &= (y^{-1}, x)\sigma. \end{aligned}$$

Below, we will use the GAP package `AutomGrp` [MS08]. For the convenience of the reader, in nontrivial cases we will provide a code used to obtain the results. We start from encoding G , together with extra generators x and y , in `AutomGrp`:

```
gap> L:=SelfSimilarGroup("a=(c,b)(1,2), b=(a,a)(1,2), c=(a,a),\
> x=(y,x^-1)(1,2), y=(y^-1,x)");
< a, b, c, x, y >
```

We first observe that the following relations hold in G (as can be verified either by hands or using `IsOne` or `FindGroupRelation` commands in `AutomGrp`):

$$(12) \quad a^4 = b^4 = c^4 = [b, c] = (cb^{-1})^2 = 1,$$

$$(13) \quad b^a b^{a^{-1}} = c^a c^{a^{-1}} = a^c a^{b^{-1}} = 1.$$

Lemma 4.2. *The derived subgroup G' of G has index 8 in G and the abelianization G/G' of G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.*

Proof. It follows from (13) that the images of generators a, b, c in the abelianization G/G' all have order 2. Thus, G/G' may have at most 8 elements and the commutator subgroup G' has index at most 8 in G . On the other hand, by looking at the third level of the tree we deduce that this index has to be at least 8:

```
gap> Size(PermGroupOnLevel(G,3));
64
gap> Size(DerivedSubgroup(PermGroupOnLevel(G,3)));
8
```

□

The Reidemeister-Schreier procedure with the system of coset representatives $T = \{1, a, b, c, ab, ac, bc, abc\}$ yields:

$$(14) \quad G' = \langle a^2, [b^{-1}, a^{-1}], [c^{-1}, a^{-1}] \rangle = \langle a^2, [a, b], [a, c] \rangle.$$

Moreover, this generating set is minimal, what can be seen already on the third level of the tree while passing to corresponding finite quotients.

Consider the subgroup H of G defined by

$$H = \langle a^2, x, y \rangle.$$

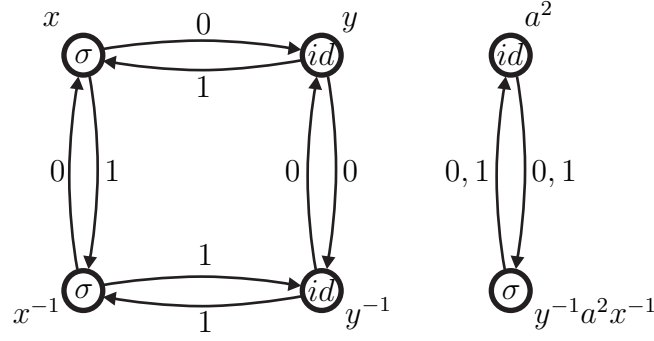
As we will use H in the computations below, we also encode it in `AutomGrp`.

```
gap> H := Group(a^2, x, y);
< a^2, x, y >
```

Proposition 4.3. *Subgroup H is a subgroup of G of index 2 (hence, H is normal in G and contains G'). Moreover, $G = \langle H, a \rangle$.*

Proof. Since $[a, b] = a^2 y^{-2}$ and $[a, c] = a^2 x^2$, by Equation (14) we get that $G' < H$. Further, since $G = \langle a, H \rangle$, in order to check that H is normal in G it is enough to check that H is closed under the conjugation by a and a^{-1} , which follows from the following identities:

$$(15) \quad \begin{aligned} x^a &= x^{-1} a^2, \\ y^a &= a^2 y^{-1}, \\ x^{a^{-1}} &= a^2 x^{-1}, \\ y^{a^{-1}} &= y^{-1} a^2. \end{aligned}$$


 FIGURE 2. Automaton generating the rank 2 lamplighter group $\mathcal{L}_{2,2}$

Further, since the permutation induced by a on the third level of the tree does not belong to the permutation group acting on the third level induced by H :

```
gap> PermOnLevel(a,3) in PermGroupOnLevel(H,3);
false
```

we get that $a \notin H$. On the other hand, since $a^2 \in H$ and H is normal in G , we get that H has index 2 in G . \square

The following proposition completely describes the structure of H , and hence, G .

Proposition 4.4. *The group $H = \langle a^2, x, y \rangle = \langle a^2, yx, y \rangle$ is isomorphic to the rank 2 lamplighter group $\mathcal{L}_{2,2} = (\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}$, where the isomorphism is induced by sending generators a^2 and yx of H to generators of $(\mathbb{Z}/2\mathbb{Z})^2$ of $\mathcal{L}_{2,2}$ and the generator $y \in H$ to the generator of \mathbb{Z} in $\mathcal{L}_{2,2}$. Moreover, H is a self-similar group generated by the automaton depicted in Figure 2.*

The strategy for the proof of this theorem is similar to the one used in [GŻ01], but is more general and involves more details. We start from an auxiliary definition.

Definition 17. *An automorphism g of the tree X^* is called spherically homogeneous if for each level l the sections of g at all vertices of X^l act identically on the first level (or, equivalently, coincide).*

Every such automorphism can be defined by a sequence $\{\sigma_n\}_{n \geq 1}$ of permutations of X where σ_n describes the action of g on the n -th letter of the input word over X . Given a sequence $(\sigma_n)_{n \geq 1}$ we will denote the corresponding spherically homogeneous automorphism by $[\sigma_n]_{n \geq 1}$ or simply as $[\sigma_1, \sigma_2, \sigma_3, \dots]$.

Obviously, all spherically homogeneous automorphisms of X^* form a group, which we denote by $\text{SHAut}(X^*)$, isomorphic to a product of uncountably many copies of $\text{Sym}(|X|)$. In the case of a binary tree, this groups is abelian and isomorphic to the abelianization of $\text{Aut}(T_2)$, which, in turn, is isomorphic to $\prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$.

Below, we will prove that G is contained in the normalizer of $\text{SHAut}(X^*)$ in $\text{Aut}(X^*)$, even though neither of generators A or B is spherically homogeneous. It is implicitly proved in [GŻ01] that standard representation of a lamplighter group in $\text{Aut}(T_2)$ is contained in the normalizer of $\text{SHAut}(X^*)$.

The following terminology is motivated by similar one in [GŻ01].

Definition 18 (Generalized Conjugations). *Let x and y be as before.*

- (a) *For an element $g \in \text{Aut}(T_2)$, by a generalized elementary conjugation of g we mean the elements $y^{-1}gy$, $y^{-1}gx^{-1}$, xgy , xgx^{-1} , ygy^{-1} , $x^{-1}gy^{-1}$, ygx and $x^{-1}gx$.*
- (b) *The first four elements in (a) are called positive elementary conjugations and the latter four elements by negative elementary conjugations.*
- (c) *A composition of k generalized (positive, negative) elementary conjugations is called a generalized (positive, negative) conjugation of length k .*

For example, $y^{-1}y^{-1}x \cdot g \cdot x^{-1}yx^{-1}$ is a generalized positive conjugation of g of length 3.

Lemma 4.5. *The generalized conjugations of spherically homogeneous automorphisms are spherically homogeneous.*

Proof. By induction on the length of a generalized conjugation, it is enough to prove the lemma only for elementary generalized conjugations.

The key observation required for the proof is that both $xy = (xy)^{-1}$ and $yx = (yx)^{-1}$ are spherically homogeneous, and, consequently, commute with each $q \in \text{SHAut}(X^*)$. Indeed, we have

$$(16) \quad \begin{aligned} xy &= [\sigma, \sigma, 1, 1, 1, \dots] \\ yx &= [\sigma, 1, 1, 1, 1, \dots] = \sigma \end{aligned}$$

Therefore, for each $q \in \text{SHAut}(X^*)$ we have $xy \cdot q = q \cdot xy$ and hence,

$$(17) \quad yqy^{-1} = x^{-1}qx.$$

Similarly, we get

$$(18) \quad \begin{aligned} y^{-1}qx^{-1} &= xqy^{-1}, \\ yqx &= x^{-1}qy^{-1}, \\ yqy^{-1} &= x^{-1}qx. \end{aligned}$$

Therefore, it is enough to consider only 4 elementary generalized conjugations of each element (2 positive and 2 negative). The statement of the lemma follows by induction on the level from the identities below and Equations (17) and (18). Consider two cases.

Case I. $q = (q', q')\sigma$. Then

$$(19) \quad \begin{aligned} y^{-1}qy &= (yq'x, x^{-1}q'y^{-1})\sigma = (yq'x, yq'x)\sigma, \\ y^{-1}qx^{-1} &= (yq'y^{-1}, x^{-1}q'x)\sigma = (yq'y^{-1}, yq'y^{-1}), \\ yqx &= (y^{-1}q'x^{-1}, xq'y)\sigma = (xq'y, xq'y), \\ yqy^{-1} &= (y^{-1}q'x^{-1}, xq'y)\sigma = (xq'y, xq'y)\sigma. \end{aligned}$$

Case II. $q = (q', q')$. Then

$$(20) \quad \begin{aligned} y^{-1}qy &= (yq'y^{-1}, x^{-1}q'x) = (yq'y^{-1}, yq'y^{-1}), \\ y^{-1}qx^{-1} &= (yq'x, x^{-1}q'y^{-1})\sigma = (yq'x, yq'x)\sigma, \\ yqx &= (y^{-1}q'y, xq'x^{-1})\sigma = (y^{-1}q'y, y^{-1}q'y)\sigma, \\ yqy^{-1} &= (y^{-1}q'y, xq'x^{-1}) = (y^{-1}q'y, y^{-1}q'y). \end{aligned}$$

Note that the sections of positive elementary generalized conjugations are negative elementary generalized conjugations and vice versa. \square

Recall that xy is spherically homogeneous. It is crucial for the arguments below that a^2 is spherically homogeneous as well (note that a is not spherically homogeneous). It is straightforward to check that

$$(21) \quad a^2 = [1, \sigma, 1, \sigma, 1, \sigma, 1, \dots],$$

where 1's and σ 's alternate with level. Therefore, by Lemma 4.5 all conjugates of a^2 and xy by powers of y are spherically homogeneous, and thus, all of them are involutions and commute with each other. To finish the proof of Proposition 4.4 it is now enough to show that all these conjugates are different.

It is proved in [BGK⁺08] that all conjugates of yx by powers of y are different and finitary (i.e. have nontrivial sections only up to some finite level). This automatically implies that $(a^2)^{y^i} \neq (yx)^{y^j}$ for any i, j . Indeed, if $(a^2)^{y^i} = (yx)^{y^j}$, then $a^2 = (yx)^{y^{j-i}}$ must be finitary, which is not the case.

Thus, it is left to show that $(a^2)^{y^i} \neq (a^2)^{y^j}$ for $i \neq j$. For this, of course it suffices to construct an infinite number of different conjugates of a^2 by powers of y .

The fact that all conjugates of yx by powers of y are different was proved in [BGK⁺08] by explicitly computing the depth of $(yx)^{y^i}$ for all i , where the depth of a finitary automorphism h is the smallest level of the tree such that all sections of h at the vertices of this level are trivial. In our case, even though the conjugates of a^2 are not finitary any more, the conjugates of $(a^2)^{y^{-1}}$ by positive powers of y^3 are “antifinitary” in the following sense.

Definition 19. *An automorphism g of T_2 is called antifinitary if there exists a level k such that the sections of g at all vertices of this level coincide with the automorphism $s = (s, s)\sigma = [\sigma, \sigma, \sigma, \dots]$ that changes all letters in any input word to the opposite ones.*

The smallest k with the above property is called the antidepth of g .

The goal of the following lemmas is to show that the conjugates of a^2 by powers of y are all different.

Lemma 4.6. *If $g \in \text{SHAut}(X^*)$ is a spherically homogeneous automorphism of T_2 , then for each $v \in X^*$ the section of a generalized elementary conjugation of g at v is a generalized elementary conjugation of $g|_v$. Moreover, the positive and the negative conjugations alternate with the level.*

Proof. For $|v| = 1$ the statement follows from Equations (19) and (20). Then the Lemma follows trivially by induction on $|v|$. \square

As a direct corollary of the above Lemma we obtain:

Corollary 4.7. *If g is a spherically homogeneous automorphism of T_2 , then for each $v \in X^*$ of even length, the section of a generalized positive conjugation of g of length k at v is a generalized positive conjugation of $g|_v$ of length k .*

Define the following antifinitary automorphisms of T_2 :

$$\begin{aligned} q &= [\sigma, \sigma, 1, \sigma, 1, 1, \sigma, \sigma, \sigma, \dots], \\ w &= [1, 1, 1, \sigma, 1, 1, \sigma, \sigma, \sigma, \dots]. \end{aligned}$$

Since g is spherically homogeneous, by Lemma 4.5 all generalized elementary conjugations of g are also spherically homogeneous. The next lemma exhibits more structure.

Lemma 4.8.

- (a) For each positive generalized conjugation h of q of length 3, and for each $v \in X^6$

$$h|_v = w.$$

- (b) For each positive generalized conjugation h of w of length 3, and for each $v \in X^6$

$$h|_v = q.$$

Proof. We use `AutomGrp` to check these identities. First, we define elements q and w in `GAP`. Since we are about to compute generalized conjugations of these elements, we will redefine the whole group G by adding q, w , and their sections to the list of generators. We note that as will be shown in Lemma 4.9, both q and w are elements of G , so since G is self-similar, we do not change the whole group by doing this. We will not use this fact in future.

```
gap> G:=SelfSimilarGroup("a=(c,b)(1,2),b=(a,a)(1,2),c=(a,a),\
> x=(y,x^-1)(1,2),y=(y^-1,x),\
> w=(w1,w1),w1=(w2,w2),w2=(w3,w3),w3=(w4,w4)(1,2),\
> w4=(w5,w5),w5=(w6,w6),w6=(w6,w6)(1,2),\
> q=(q1,q1)(1,2),q1=(q2,q2)(1,2),q2=(q3,q3),q3=(q4,q4)(1,2),\
> q4=(q5,q5),q5=(q6,q6),q6=(q6,q6)(1,2)");
< a, b, c, x, y, w, w1, w2, w3, w4, w5, w6, q, q1, q2, q3,\
q4, q5, q6 >
```

There are only 2 different generalized positive elementary conjugations of each element (recall Equations (17) and (18)). Therefore, there are 8 potentially different generalized positive elementary conjugations of length 3. Below, we verify the statement of the lemma by checking all eight possible cases.

For (a) we have:

```
gap> Section(y^-3*q*y^3,[1,1,1,1,1,1])=w;
true
gap> Section(y^-3*q*y^2*x^-1,[1,1,1,1,1,1])=w;
true
gap> Section(y^-3*q*y*x^-1*y,[1,1,1,1,1,1])=w;
true
gap> Section(y^-3*q*y*x^-2,[1,1,1,1,1,1])=w;
true
gap> Section(y^-3*q*x^-1*y^2,[1,1,1,1,1,1])=w;
true
gap> Section(y^-3*q*x^-1*y*x^-1,[1,1,1,1,1,1])=w;
true
gap> Section(y^-3*q*x^-2*y,[1,1,1,1,1,1])=w;
true
gap> Section(y^-3*q*x^-3,[1,1,1,1,1,1])=w;
true
```

Similarly for (b):

```

gap> Section(y^-3*w*y^3,[1,1,1,1,1,1])=q;
true
gap> Section(y^-3*w*y^2*x^-1,[1,1,1,1,1,1])=q;
true
gap> Section(y^-3*w*y*x^-1*y,[1,1,1,1,1,1])=q;
true
gap> Section(y^-3*w*y*x^-2,[1,1,1,1,1,1])=q;
true
gap> Section(y^-3*w*x^-1*y^2,[1,1,1,1,1,1])=q;
true
gap> Section(y^-3*w*x^-1*y*x^-1,[1,1,1,1,1,1])=q;
true
gap> Section(y^-3*w*x^-2*y,[1,1,1,1,1,1])=q;
true
gap> Section(y^-3*w*x^-3,[1,1,1,1,1,1])=q;
true

```

This finishes the proof. \square

Lemma 4.9. *For each $i \geq 1$ and $v \in X^{12i-8}$, we have $(a^2)^{y^{6i-1}}|_v = q$.*

Proof. We proceed by induction on i . For $i = 1$ we have:

```

gap> Section((a^2)^(y^5),[1,1,1,1])=q;
true

```

The induction step follows from Lemmas 4.6 and 4.8. Indeed, suppose $(a^2)^{y^{6i-1}}|_v = q$ for some i and vertex $v = 1^{12i-8} \in X^{12i-8}$ (recall that by Lemma 4.5 all conjugates of a^2 by powers of y are spherically homogeneous, so the section does not depend on the choice of v in X^{12i-8}). Then by Corollary 4.7 $(a^2)^{y^{6i-1+3}}|_v$ is a positive (since $12i - 8$ is even) generalized conjugation h of length 3 of q . Thus, by Lemma 4.8 (a)

$$(a^2)^{y^{6i-1+3}}|_{v1^6} = ((a^2)^{y^{6i-1}})^{y^3}|_v|_{1^6} = h|_{1^6} = w.$$

Repeating the same argument one more time and applying Lemma 4.8(b) yields

$$(a^2)^{y^{6i-1+6}}|_{v1^{12}} = (a^2)^{y^{6(i+1)-1}}|_{1^{12(i+1)-8}} = q,$$

which finishes the proof. \square

Corollary 4.10. *For each $i \geq 1$ the antidepth of $(a^2)^{y^{6i-1}}$ is equal to $12i - 2$. In particular, all conjugates of a^2 by powers of y are different.*

Proof. The first part immediately follows from Lemma 4.9 and the fact that $(a^2)^{y^i}$ is spherically homogeneous by Lemma 4.5. Furthermore, if $(a^2)^{y^i} = (a^2)^{y^j}$ for some $i \neq j$, then there could be at most $|i - j|$ different conjugates of a^2 by powers of y , which contradicts to the first part. \square

Now we have all the ingredients to prove Proposition 4.4.

Proof of Proposition 4.4. We have already shown above that $(a^2)^{y^i}$ and $(yx)^{y^i}$, $i \in \mathbb{Z}$ all commute and have order 2. As was already mentioned, it was proved in [BGK⁺08] (automaton 891) that $L = \langle x, y \rangle$ is isomorphic to the lamplighter group and that

$(yx)^{y^i}$ are all different and finitary. Corollary 4.10 guarantees that $(a^2)^{y^i}$ are distinct for all $i \in \mathbb{Z}$. So it remains to show that $(a^2)^{y^i}$ is not in L for each $i \in \mathbb{Z}$. Since for each i the order of $(a^2)^{y^i}$ is 2 (because it is a spherically homogeneous automorphism), this element could potentially be equal only to an element of the base group in L isomorphic to $\oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ (because these are the only elements in the lamplighter group of order 2), i.e., an element of the form $(yx)^{y^j}$. But, as indicated above, this is not possible since in this case a^2 would be finitary, which is not the case.

Thus, the group $\langle (a^2)^{y^i}, (yx)^{y^j}, i, j \in \mathbb{Z} \rangle$ is isomorphic to an infinite direct product of countably many copies of $(\mathbb{Z}/2\mathbb{Z})^2$. The infinite cyclic group $\langle y \rangle$ acts on this product by conjugation, that corresponds to simply shifting the exponent of y . Consequently, the group $H = \langle a^2, x, y \rangle$ has a structure of the rank 2 lamplighter group

$$H \cong \mathcal{L}_{2,2} = (\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}.$$

□

Now we can proceed to the proof of the main theorem of this subsection.

Proof of Theorem 4.1. First of all, note that since metabelian group H is a normal subgroup of index 2 in G , the group G itself has a derived length at most 3. On the other hand, since $[[a, b], [a, c]] \neq 1$:

```
gap> IsOne(Comm(Comm(a,b),Comm(a,c)));
false
```

the group G cannot be metabelian and hence has derived length 3.

Recall that $G = \langle H, a \rangle$, the element a has order 4 and $a^2 \in H$. Therefore G is not a semidirect product of H and $\langle a \rangle$. However, the element $t = ax^{-1} = ac^{-1}a$ has order 2 and is certainly not in H as $a \notin H$ and $x \in H$. Therefore,

$$G = H \rtimes \langle t \rangle \cong ((\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where the action of t on generators of H is defined as

$$(22) \quad \begin{aligned} x^t &= (x^a)^{x^{-1}} = (x^{-1}a^2)^{x^{-1}} = a^2x^{-1}, \\ y^t &= (y^a)^{x^{-1}} = (a^2y^{-1})^{x^{-1}} = xa^2y^{-1}x^{-1}, \\ (a^2)^t &= ((a^2)^a)^{x^{-1}} = (a^2)^{x^{-1}} = xa^2x^{-1}, \end{aligned}$$

as follows from Equation (15). Taking into account that $b^{-1}c = yx$ and $b^{-1}a = y$ produces equalities (9).

To get a presentation for G , we start from a presentation of G coming from its structural description described above. Let $\xi = a^2$, $\eta = yx = b^{-1}c$, $y = b^{-1}a$ and $t = ac^{-1}a$ be the generators of G . Then $\mathcal{L}_{2,2} = \langle \xi, \eta, y \rangle \triangleleft G$ has the following presentation as a rank 2 lamplighter group:

$$\mathcal{L}_{2,2} = \langle \xi, \eta, y \mid \xi^2 = \eta^2 = 1, [\xi, \xi^{y^i}] = [\xi, \eta^{y^i}] = [\eta, \eta^{y^i}] = 1, i \geq 1 \rangle.$$

The action of t on generators of $\mathcal{L}_{2,2}$ follows from equations (22) and the identity $x = y^{-1}\eta$.

$$(23) \quad \xi^t = (a^2)^t = xa^2x^{-1} = y^{-1}\eta\xi\eta^{-1}y,$$

$$(24) \quad \eta^t = (yx)^t = xa^2y^{-1}x^{-1} \cdot a^2x^{-1} = y^{-1}\eta\xi y^{-1}\eta^{-1}y \cdot \xi\eta^{-1}y,$$

$$(25) \quad y^t = xa^2y^{-1}x^{-1} = y^{-1}\eta\xi y^{-1}\eta^{-1}y.$$

Therefore the presentation for G with respect to generators ξ, η, y and t is

$$(26) \quad G = \langle \xi, \eta, y, t \mid \xi^2 = \eta^2 = 1, [\xi, \xi^{y^i}] = [\xi, \eta^{y^i}] = [\eta, \eta^{y^i}] = 1, i \geq 1, \\ t^2 = 1, \quad \xi^t = y^{-1}\eta\xi\eta^{-1}y, \\ \eta^t = y^{-1}\eta\xi y^{-1}\eta^{-1}y\xi\eta^{-1}y, \quad y^t = y^{-1}\eta\xi y^{-1}\eta^{-1}y \rangle.$$

To finish the proof we only need to rewrite presentation (26) in terms of generators a, b and c . The relation in the first line of (26) are rewritten simply by substituting $\xi = a^2, \eta = b^{-1}c, y = b^{-1}a$. These relations correspond precisely to the relations in the first two lines in the presentation (10).

The relation $t^2 = (ac^{-1}a)^2 = 1$ is equivalent to

$$(27) \quad (ca^2)^2 = 1$$

taking into account that $a^4 = 1$.

Further, relation (23) yields

$$(a^2)^{ac^{-1}a} = a^{-1}b \cdot b^{-1}c \cdot a^2 \cdot c^{-1}a = a^{-1}ca^2c^{-1}a,$$

that trivially holds in a free group.

Relation (25) is equivalent to

$$(b^{-1}a)^{ac^{-1}a} = a^{-1}b \cdot b^{-1}c \cdot a^2 \cdot a^{-1}b \cdot c^{-1}b \cdot b^{-1}a = a^{-1}ca^{-1} \cdot a^2ba^{-1} \cdot ac^{-1}a,$$

that simplifies to $b^{-1}a = a^2ba^{-1}$ or, equivalently, to

$$(28) \quad (ba^2)^2 = 1.$$

Finally, relation (24) is equivalent to

$$(b^{-1}c)^{ac^{-1}a} = a^{-1}b \cdot b^{-1}c \cdot a^2 \cdot a^{-1}b \cdot c^{-1}b \cdot b^{-1}a \cdot a^2 \cdot c^{-1}b \cdot b^{-1}a = \\ a^{-1}ca^{-1} \cdot a^2bc^{-1}a^2 \cdot ac^{-1}a,$$

which again simplifies to $bc^{-1} = a^2bc^{-1}a^2$ and now follows trivially from relations (27) and (28). This finishes the proof of the theorem. \square

Proposition 4.11. *The automorphism ζ of a free group $F(a, b, c)$ defined by $\zeta(a) = a, \zeta(b) = c, \zeta(c) = b$ induces an automorphism of G .*

Proof. It is obvious that the images of relators in the first and the third lines of presentation (10) of G under ζ are again relators in G . To see that ζ sends relators in the second line of (10) to the identity element of G it is enough to notice that $c^{-1}a = (c^{-1}b) \cdot (b^{-1}a)$ and that $c^{-1}b = (b^{-1}c)^{-1}$ commutes with conjugates of a^2 and $(b^{-1}c)$ by powers of $b^{-1}a$. Indeed, we first prove by induction that

$$(a^2)^{(c^{-1}a)^i} = (a^2)^{(b^{-1}a)^i}$$

for all $i \geq 0$. For $i = 0$ there is nothing to prove; the induction step is proved as follows:

$$(a^2)^{(c^{-1}a)^{i+1}} = ((a^2)^{(c^{-1}a)^i})^{c^{-1}a} = \left(((a^2)^{(b^{-1}a)^i})^{c^{-1}b} \right)^{b^{-1}a} = (a^2)^{(b^{-1}a)^{i+1}}.$$

The same argument is also used to show that for all $i \geq 1$

$$(b^{-1}c)^{(c^{-1}a)^i} = (b^{-1}c)^{(b^{-1}a)^i}.$$

Therefore, for the relators in the second line of (10) we have:

$$\begin{aligned} \zeta \left(\left[a^2, (a^2)^{(b^{-1}a)^i} \right] \right) &= \left[a^2, (a^2)^{(c^{-1}a)^i} \right] = \left[a^2, (a^2)^{(b^{-1}a)^i} \right] = 1 \\ \zeta \left(\left[a^2, (b^{-1}c)^{(b^{-1}a)^i} \right] \right) &= \left[a^2, (b^{-1}c)^{(c^{-1}a)^i} \right] = \left[a^2, (b^{-1}c)^{(b^{-1}a)^i} \right] = 1 \\ \zeta \left(\left[b^{-1}c, (b^{-1}c)^{(b^{-1}a)^i} \right] \right) &= \left[b^{-1}c, (b^{-1}c)^{(c^{-1}a)^i} \right] = \left[b^{-1}c, (b^{-1}c)^{(b^{-1}a)^i} \right] = 1 \end{aligned}$$

Therefore, ζ induces an endomorphism of G . This endomorphism is obviously onto and also one-to-one since ζ is an involution. \square

Corollary 4.12. *The group G acts essentially freely on the boundary of the tree.*

Proof. The stabilizer of the first level in G is generated by

$$\begin{aligned} b^{-2}cbcb^{-1}c &= (a, a), \\ cb^{-1}a &= (b, c), \\ ac^{-1}b^{-1}c^2 &= (c, b). \end{aligned}$$

In this situation Proposition 4.11 guarantees that we can apply Proposition 3.5 and deduce that the action of G on the boundary of the tree is essentially free. \square

We end up this section with the following interesting observations.

Proposition 4.13. *The group $A = \langle (yx)^{y^i}, i \in \mathbb{Z} \rangle$ coincides with a group of all finitary spherically homogeneous automorphisms.*

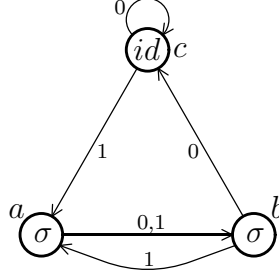
Proof. It is proved in [BGK⁺08] (see automaton 891) that all elements of the form $s_n = (yx)^{y^{-n}}$ are finitary spherically homogeneous automorphisms with depth $2n+1$ for nonnegative n and $2(-n)$ for negative n . The propositions now immediately follow by induction on the level. \square

Proposition 4.14.

- (a) *The subgroup $L = \langle x, y \rangle$ of G has infinite index in G .*
- (b) *The closure \bar{L} of L has index 2 in the closure \bar{G} of G .*

Proof. (a) According to Theorem 3.5 in [GK12] each subgroup of $\mathcal{L}_{2,2} \cong H$ of finite index must be isomorphic to $\mathcal{L}_{2,s}$ for some $s \geq 1$. Since $L = \langle x, y \rangle$ is isomorphic to a standard lamplighter group \mathcal{L} , it cannot have a finite index in H , and thus in G .

(b) By proposition 4.13 the group of all spherically homogeneous automorphisms coincides with the closure \bar{A} of A , where A is from Proposition 4.13. Thus, as by Equality (21) a^2 is spherically homogeneous, $a^2 \in \bar{A} < \bar{L}$. Therefore, $H = \langle a^2, x, y \rangle < \bar{L}$ and $\bar{H} < \bar{L}$. On the other hand, $L < H$ and so $\bar{L} < \bar{H}$ and $\bar{L} = \bar{H}$. Since H has index 2 in G , $\bar{L} = \bar{H}$ has index at most 2 in \bar{G} . Finally, since a induces a permutation of the third level of the tree that does not belong to the permutation group on this level induced by H , we must have that $a \notin \bar{H}$. Thus, $\bar{L} = \bar{H}$ has index 2 in \bar{G} . \square


 FIGURE 3. Automaton \mathcal{A}_{2372} generating G_{2372}

4.4. Automaton 2372. Throughout this subsection let G denote the group G_{2372} generated by automaton \mathcal{A}_{2372} and defined by the following wreath recursion: $a = (b, b)\sigma, b = (c, a)\sigma, c = (c, a)$. The automaton itself is shown in Figure 3. We start from stating the main theorem of this section that will be the ground for the proof of essential freeness of the action of G on ∂T_2 .

Theorem 4.15. *The group G is solvable of derived length 3 and has the following structure:*

$$G = (K \rtimes \langle v \rangle) \rtimes \langle a \rangle \cong \left(\frac{1}{2}\mathbb{Z} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rtimes (\mathbb{Z}/2\mathbb{Z}) \right) \rtimes \mathbb{Z}$$

where K is isomorphic to $\frac{1}{2}\mathbb{Z} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, v acts on K by taking each element to its inverse when K is identified with $\frac{1}{2}\mathbb{Z} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, the action of a generator a of \mathbb{Z} on K corresponds to the multiplication by 3, and $v^a = vx_0, v^{a^{-1}} = vx_1^{-1}$ with x_i being the image of 3^{-i} under an isomorphism $\frac{1}{2}\mathbb{Z} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow K$.

Moreover, G has the following finite presentations

$$(29) \quad G \cong \langle a, b, c \mid (ac^{-1})^a = (ac^{-1})^3, (ab^{-1})^2 = 1, (ac^{-1})^{ab^{-1}} = ca^{-1}, b^{-1}a = ab^{-1}(ac^{-1})^2 \rangle,$$

where a, b, c are generators of G corresponding to the states of automaton \mathcal{A}_{2372} .

The proof will be elaborated through the sequence of lemmas and propositions below.

Let $x_0 = (ac^{-1})^2$ and $x_{n+1} = x_n^{a^{-1}} = x_0^{a^{-n-1}}$ be elements of G . Note that x_0 has infinite order since it acts spherically transitively on the levels of the tree (that can be checked by computing its image in the abelianization of $\text{Aut}(X^*)$ using, for exampl, package `automgrp`). Further,

$$x_{n+1}^3 = x_n, \quad n \geq 0.$$

Indeed,

$$x_{n+1}^3 = (x_0^{a^{-n-1}})^3 = (x_0^3)^{a^{-n-1}} = ((x_0^3)^{a^{-1}})^{a^{-n}} = x_0^{a^{-n}} = x_n,$$

since $(x_0^3)^{a^{-1}} = x_0$. This, in particular, implies that x_n are all different.

By the above argument the subgroup $H = \langle x_0, x_1, x_2, \dots \rangle$ of G is isomorphic to an additive group $\mathbb{Z} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ of all rational numbers whose denominators are powers of 3 via an isomorphism that sends x_i to 3^{-i} . In particular, H is abelian.

Lemma 4.16. *The subgroup H of G is normal in G .*

Proof. By definition of x_n we have $x_n^a = x_{n-1}$ for $n > 0$ and $x_n^{a^{-1}} = x_{n+1}$ for $n \geq 0$. Also, one can check that $x_0^a = x_0^3$, so $H^a = H$.

Next, we show that $H^c = H$. It is straightforward to check that $x_0^c = x_0^3$ and that $x_0^{c^{-1}} = x_1$. We will now show that $x_n^c = x_n^a = x_{n-1}$ for $n \geq 1$ by induction on n . The base case $n = 0$ was mentioned above. Suppose that $x_n^c = x_n^a$ is true for some n . Then

$$\begin{aligned} ((x_0)^{a^{-n}})^c &= ((x_0)^{a^{-n}})^a \\ ((x_0)^{a^{-n+1}})^{a^{-1}c} &= (x_0)^{a^{-n+1}} \end{aligned}$$

and thus

$$x_{n-1}^{a^{-1}c} = x_{n-1}$$

or

$$(30) \quad [x_{n-1}, a^{-1}c] = 1$$

Now consider the equality $x_{n+1}^c = x_{n+1}^a$ that we are going to prove. Similarly as above this equality is equivalent to

$$(31) \quad [x_n, a^{-1}c] = 1.$$

Since $x_n = x_{n-1}^{a^{-1}}$ we can rewrite (31) as

$$\begin{aligned} [x_{n-1}^{a^{-1}}, a^{-1}c] &= 1 \\ [x_{n-1}, (a^{-1}c)^a]^{a^{-1}} &= 1. \end{aligned}$$

Finally, the last equality holds true because of the inductive assumption (30) and relation $(a^{-1}c)^a = (a^{-1}c)^3$ in G .

Now one can simply mention that $x_n^c = x_{n-1}$ implies $x_n^{c^{-1}} = x_{n+1}$ for $n \geq 0$. Thus we get $H^c = H$.

The last step for normality of H in G is to show that $H^b = H$. We proceed similarly to the previous case. First, note that $x_0^b = x_0^{-3}$. Now we will prove by induction on n that

$$x_n^b = (x_n^a)^{-1} = x_{n-1}^{-1}$$

for $n > 0$, and, equivalently,

$$x_n^{b^{-1}} = (x_n^{a^{-1}})^{-1} = x_{n+1}^{-1}$$

for $n \geq 0$. The above equalities hold true for $n = 0$. Further, assume that $x_k^b = (x_k^a)^{-1}$ is true for all $k \leq n$. Then, by induction assumption, we have:

$$x_{n-1}^{ba^{-1}b^{-1}a} = (x_{n-1}^{-3})^{a^{-1}b^{-1}a} = (x_{n-1}^{-1})^{b^{-1}a} = x_n^a = x_{n-1}.$$

In other words

$$[x_{n-1}, [b^{-1}, a]] = 1.$$

It is straightforward to verify the following identity in G :

$$(32) \quad [b^{-1}, a]^a = [b^{-1}, a]^3.$$

Using this identity we get

$$[x_n, [b^{-1}, a]]^a = [x_{n-1}^{a^{-1}}, [b^{-1}, a]]^a = [x_{n-1}, [b^{-1}, a]^a] = [x_{n-1}, [b^{-1}, a]^3] = 1,$$

and thus

$$[x_n, [b^{-1}, a]] = 1.$$

The last identity can be rewritten as $x_n^{ba^{-1}b^{-1}a} = x_n$ or $(x_n^b)^{a^{-1}b^{-1}} = x_n^{a^{-1}} = x_{n+1}$. Applying inductive assumption we obtain

$$x_{n+1} = (x_{n-1}^{-1})^{a^{-1}b^{-1}} = (x_n^{-1})^{b^{-1}}$$

and, finally,

$$x_n^{b^{-1}} = x_{n+1}^{-1}.$$

□

Proposition 4.17. *The group G is metabelian. Moreover, $G' = H$.*

Proof. By inspection one can check that

$$(33) \quad [a, b] = [b, c] = [c, a] = x_0^{-3}$$

and

$$[ab, ba^{-1}] = x_0^2.$$

Therefore, $x_0 \in G'$ and, hence, $H < G'$ as H is generated by x_0 as normal subgroup of G . On the other hand, equality (33) and Lemma 4.16 show that $G' = \langle [a, b], [a, c], [b, c] \rangle^G < H$. Thus $G' = H$ and hence G is metabelian. □

Corollary 4.18. *The group G has exponential growth.*

Proof. This follows from Rosset's theorem (see [Ros76]) that states that if a finitely generated group G which is not of exponential growth contains a normal subgroup A with G/A solvable, then A is finitely generated. In our case, $H = G' \cong \mathbb{Z} \left[\frac{1}{3} \right]$ is not finitely generated and G/H is abelian, thus G must have exponential growth. □

Lemma 4.19. *The centralizer $C_G(H)$ of H in G is generated by H and $t = ac^{-1}$.*

Proof. The action of G on H by conjugation induces a homomorphism

$$\phi: G \rightarrow \text{Aut}(H).$$

The kernel of ϕ is precisely $C_G(H)$.

For a freely reduced word w in a, b and c and $s \in \{a, b, c\}$ let $\exp_s(w)$ denote the exponent sum of the generator s in w . Also denote

$$\exp(w) = \exp_a(w) + \exp_b(w) + \exp_c(w).$$

Since from Lemma 4.16 we know that

$$(34) \quad \begin{aligned} x_n^a &= x_{n-1} = x_n^3, \\ x_n^b &= x_{n-1}^{-1} = x_n^{-3}, \\ x_n^c &= x_{n-1} = x_n^3, \end{aligned}$$

for $n \geq 1$ and $x_0^a = x_0^c = x_0^3, x_0^b = x_0^{-3}$. In other words a and c act on $H \cong \mathbb{Z} \left[\frac{1}{3} \right]$ by multiplying each element by 3 and b acts by multiplying each element by -3 . Thus, we can write

$$C_G(H) = \{w(a, b, c) \mid \exp(w) = 0, \exp_b(w) \text{ is even}\},$$

where $w(a, b, c)$ denotes the word in a, b and c evaluated in G .

This description implies, in particular, that $t = ac^{-1} \in C_G(H)$ and that

$$K := \langle H, t \rangle$$

is a subgroup of $C_G(H)$ (since H is abelian). Let us prove the converse inclusion. Suppose $w = w(a, b, c) \in C_G(H)$, so that $\exp(w) = 0$ and $\exp_b(w)$ is even. Then using the identity $gh = hg[g, h]$ and the fact that by Lemma 4.16 $H = G'$, we can write $w = w'h$ for a word $w' \in C_G(H)$ and $h \in H$, where all occurrences of b in w' are in the subwords of the form $a^{-2}b^2$, $c^{-2}b^2$ or their inverses. But these subwords are elements of H since

$$\begin{aligned} a^{-2}b^2 &= (ac^{-1})^{-6} \in H, \\ c^{-2}b^2 &= (ac^{-1})^6 \in H. \end{aligned}$$

Hence, we can write that $w = w''h$ for w'' that does not involve b . Finally, again using the fact that $G' = H$ and that

$$\exp_a(w) + \exp_c(w) = \exp(w) = 0$$

we can rewrite w'' in the form $(ac^{-1})^r h'$ for some $h' \in H$ and $r \in \mathbb{Z}$, thus obtaining a decomposition $w = (ac^{-1})^r \cdot h'h$ that shows that $w \in K = \langle H, t \rangle$. Thus $C_G(H) = K$. Also note that $t^2 = x_0$, so t does not belong to H because $\frac{1}{2} \notin \mathbb{Z}[\frac{1}{3}]$. \square

Let $\frac{1}{2}\mathbb{Z}[\frac{1}{3}]$ be an additive group of rational numbers whose denominators have form $2^k \cdot 3^l$ for $k = 0, 1$ and $l \geq 0$. Define a map $\psi: \frac{1}{2}\mathbb{Z}[\frac{1}{3}] \rightarrow C_G(H)$ by

$$\psi\left(\frac{m}{2^k 3^l}\right) = t^k x_l^m.$$

Lemma 4.20. *The map ψ is an isomorphism and $C_G(H) \cong \frac{1}{2}\mathbb{Z}[\frac{1}{3}]$.*

Proof. Follows immediately from the facts that $C_G(H)$ is abelian, $H \cong \mathbb{Z}[\frac{1}{3}]$ and $t^2 = x_0$. Note that $\psi(\frac{1}{2}) = t$ as expected. \square

Lemma 4.21. *The centralizer $C_G(H)$ is a normal subgroup of G .*

Proof. The group G acts on $C_G(H)$ by conjugation. The action on the generators of H is described above and the action on t is as follows:

$$(35) \quad \begin{aligned} t^a &= t^3, & t^{a^{-1}} &= tx_1^{-1}, \\ t^b &= t^{-3}, & t^{b^{-1}} &= x_1 t^{-1}, \\ t^c &= t^3, & t^{c^{-1}} &= tx_1^{-1}, \end{aligned}$$

This action agrees with the action on H in the sense that a and c act on $K \cong \frac{1}{2}\mathbb{Z}[\frac{1}{3}]$ by multiplying each element by 3 and b acts by multiplying each element by -3 . \square

Lemma 4.22. *The factor group G/K is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.*

Proof. Let $w \in G$ be arbitrary element. Since $G' < K$ there exists $k_1 \in K$ such that $w = a^n c^m b^l k_1$. Further, since $t = ac^{-1} \in K$ we get that

$$w = a^{n+m} a^{-m} c^m b^l k_1 = a^{n-m} (ac^{-1})^{-m} b^l k_2 = a^{n-m} b^l k_3$$

for some $k_2, k_3 \in K$. Further, since $a^{-2}b^2 = (ac^{-1})^{-6} \in K$ there exist $k \in H$, $n' \in \mathbb{Z}$ and $l' \in \{0, 1\}$ such that

$$w = a^{n'} b^{l'}.$$

Thus the set $R = \{a^n (a^{-1}b)^l \mid n \in \mathbb{Z}, l \in \{0, 1\}\}$ contains representatives of all cosets of K in G . Moreover, if $a^n K = a^m K$ for some m, n , then $a^{n-m} \in K$, which implies that $n = m$ because K is abelian and $x_0^{a^{n-m}} = x_0$ if and only if $n = m$. Similarly one shows that all elements of R represent distinct cosets of K in G .

Finally, the multiplication of the cosets is defined by

$$[a^n(ab^{-1})^l] \cdot [a^{n'}(ab^{-1})^{l'}] = [a^{n+n'}(ab^{-1})^{(l+l') \bmod 2}]$$

for $n, n' \in \mathbb{Z}$ and $l, l' \in \{0, 1\}$, since $(ab^{-1})^2 = 1$. Here for $z \in G$ we denote by $[z]$ the coset of K in G containing z .

Thus $G/K \cong \langle [a], [ab^{-1}] \rangle \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$. \square

Proposition 4.23. *The abelianization $G' = G/H$ of G is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.*

Proof. Similarly to the proof of Lemma 4.22 using the fact that $(ac^{-1})^2 = x_0 \in H$ one can show that the set $T = \{a^n(ab^{-1})^l(ac^{-1})^t \mid n \in \mathbb{Z}, l, t \in \{0, 1\}\}$ is a transversal of H in G . The multiplication of the cosets is defined by

$$[a^n(ab^{-1})^l(ac^{-1})^t] \cdot [a^{n'}(ab^{-1})^{l'}(ac^{-1})^{t'}] = [a^{n+n'}(ab^{-1})^{(l+l') \bmod 2}(ac^{-1})^{(t+t') \bmod 2}]$$

for $n, n' \in \mathbb{Z}$ and $l, l', t, t' \in \{0, 1\}$, since $(ab^{-1})^2 = 1$ and $(ac^{-1})^2 \in H$, where for $z \in G$ we denote here by $[z]$ the coset of H in G containing z .

Thus $G/H \cong \langle [a], [ab^{-1}], [ac^{-1}] \rangle \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. \square

Let $v := ab^{-1}$ be an element of G . It is easy to verify that v is an involution. Define

$$M = \langle K, v \rangle.$$

Proposition 4.24. *The group M is normal in G and has a structure $M = K \rtimes \langle v \rangle$, where v acts on K by taking each element to its additive inverse.*

Proof. Since K is normal in G for normality of M it is enough to check that $v^s \in M$ for $s \in \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$. This is indeed the case as shown below:

$$(36) \quad \begin{aligned} v^a &= vx_0, & v^{a^{-1}} &= vx_1^{-1}, \\ v^b &= vx_0, & v^{b^{-1}} &= vx_1, \\ v^c &= vx_0^{-2}, & v^{c^{-1}} &= vx_1^2. \end{aligned}$$

Furthermore, K is normal in M of index 2 and such that K and $\langle v \rangle \cong \mathbb{Z}/2\mathbb{Z}$ together generate M . Since $v \notin K$ (v does not commute with x_0 and K is abelian) we obtain that $M = K \rtimes \langle v \rangle$. It follows now from equations (34) and (35) that v acts on K by taking each element to its additive inverse. \square

Now we are ready for the proof of the main theorem of this section.

Proof of Theorem 4.15. We have that M and $\langle a \rangle$ together generate the whole group G , M is normal in G and no nontrivial power of a belongs to M . Indeed, if $a^n \in M$, then M would be of finite index in G , which is not the case since K is of index 2 in M by Proposition 4.24 and K is of infinite index in G by Lemma 4.22. Thus $G = M \rtimes \langle a \rangle$. The action of a on M and the corresponding presentation of G is obtained from relations (34), (35) and (36).

The structure of G as an iterated semidirect product yields the following infinite presentation.

$$(37) \quad G \cong \left\langle t, v, a, x_0, x_1, x_2, \dots \left| \begin{array}{lll} [x_i, t] = 1, & t^2 = x_0, & x_{i+1}^3 = x_i, \\ v^2 = 1, & x_i^v = x_i^{-1}, & t^v = t^{-1}, \\ x_i^a = x_i^3, & t^a = t^3, & v^a = vx_0, \end{array} \right. \text{ for } i, j \geq 0 \right\rangle$$

The first line of relators in the presentation (37) corresponds to the presentation of $K = \frac{1}{2}\mathbb{Z} \left[\frac{1}{3} \right]$, the second line to the presentation of $\langle v \rangle$ and its action on K , and finally the third line corresponds to the action of $\langle a \rangle$ on $M = K \rtimes \langle v \rangle$.

In order to prove that G has presentation (29) we only need to prove that all relations in (37) can be deduced from presentation (29) and vice versa. Here we give a proof of a more complicated statement that (29) implies (37) and leave the converse as exercise.

First, we note that relation $t^2 = x_0$ follows simply from the definitions of $t = ac^{-1}$ and $x_0 = (ac^{-1})^2$. The relations $t^a = t^3$, $v^2 = 1$, $t^v = t^{-1}$ and $v^a = vx_0$ precisely correspond to relations in (29).

All relations of the form $x_{i+1}^3 = x_i$ follow from the definition of $x_i = x_0^{a^{-i}}$ and the relation $x_0^a = x_0^3$ as shown in the proof of Lemma 4.16, which, in turn, follows from the definition of x_0 and the relation $t^a = t^3$. At the same time we get $x_i^3 = x_{i-1} = x_i^a$ for $i \geq 1$ again by definition of x_i , and for $i = 0$ the relation $x_0^3 = x_0^a$ is a direct consequence of $t^3 = t^a$.

The relation $[x_0, t] = 1$ holds by the definition of x_0 and t and for $i \geq 1$ we have

$$[x_i, t] = [x_{i-1}, t^a]^{a^{-1}} = [x_{i-1}, t^3]^{a^{-1}} = 1$$

by induction on i , since $t^3 = t^a$.

The last family of relations has form $x_i^v = x_i^{-1}$. We will prove that it follows from relations in (29) by induction on i . For $i = 0$ it follows from $x_0 = t^2$ and $t^v = t^{-1}$. Now assume that $x_i^v = x_i^{-1}$. Then, using the last relation in (29) and the relation $[x_i, t] = 1$, we get

$$x_i^{b^{-1}a} = x_i^{ab^{-1}(ac^{-1})^2} = (x_i^{-1})^{(ac^{-1})^2} = x_i^{-1}.$$

Thus,

$$x_{i+1}^v = (x_i^{a^{-1}})^{ab^{-1}} = x_i^{b^{-1}} = (x_i^{b^{-1}a})^{a^{-1}} = (x_i^{-1})^{a^{-1}} = x_{i+1}^{-1}.$$

□

Similarly to Proposition 4.11 we obtain the following:

Proposition 4.25. *The automorphism η of a free group $F(a, b, c)$ defined by $\eta(a) = c$, $\eta(b) = b$, $\eta(c) = a$ induces an automorphism of G .*

Proof. It is straightforward to verify that the images of relators in the presentation (29) of G under η are again relators in G . Therefore, η induces an endomorphism of G . This endomorphism is obviously onto and also one-to-one since η is an involution. □

Corollary 4.26. *The group G acts essentially freely on the boundary of the tree.*

Proof. The stabilizer of the first level in G is generated by

$$\begin{aligned} b^{-1}c^2b^{-1}c &= (a, c), \\ ab^{-1}c &= (b, b), \\ c &= (c, a). \end{aligned}$$

As in Corollary 4.12, in this situation Proposition 4.25 guarantees that we can apply Proposition 3.5 and deduce that the action of G on the boundary of the tree is essentially free. □

This subsection treated the last case in the proof of Theorem 1.1, thus finishing its proof.

5. OPEN QUESTIONS.

We end our paper with a list of some open questions.

Question 1. *Is there a group generated by finite automaton that acts neither essentially freely, nor totally non-freely on the boundary of a rooted tree?*

Question 2. *Does the total non-freeness of an action of a group generated by finite automaton on ∂T imply weak branchiness? Observe, that the converse is true [BG02, Gri11].*

Question 3. *Classify all $(4, 2)$ -groups and $(2, 3)$ -groups that act essentially freely on the boundaries of corresponding rooted trees.*

Question 4. *Are there groups generated by finite automata acting essentially freely on the boundary of rooted tree that are scale-invariant groups and are not based on the use of lamplighter type groups and groups considered in [Nek05]? (See Corollary 3.8 for motivation).*

Question 5. *Is there a hereditary just-infinite group generated by finite automaton? (See Proposition 3.7 for motivation). Note that any such group will also be an answer to Question 4.*

Question 6. *Is there a nonamenable self-replicating group?*

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